

# 功能梯度板固有频率优化的光滑有限元法研究

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**摘要:** 指数型功能梯度板的固有频率优化对提升结构动力学性能至关重要, 本文提出一种融合光滑有限元法与代理模型的高效精确优化方法. 基于一阶剪切变形理论, 构建了功能梯度板自由振动分析的光滑有限元法. 在计算系统刚度矩阵时, 对光滑域内的弯曲应变进行梯度光滑操作, 不仅避免了形函数导数的计算, 而且提高了分析精度. 通过在单元边中点独立插值剪切应变分量, 弱化位移场与剪切应变的耦合, 有效消除了剪切自锁现象. 为实现对梯度指数的优化, 首先采样并利用光滑有限元法计算样本点固有频率. 随后, 基于样本数据建立梯度指数与固有频率的代理模型, 并采用黄金分割法优化梯度指数, 使目标频率达到预设值. 数值算例结果表明, 基于分段三次 Hermite 插值的代理模型精度高, 并且结合代理模型的优化策略显著减少了调用光滑有限元法的次数, 大幅提升了优化效率, 为指数型功能梯度板固有频率优化提供了高效实用的解决方案.

**关键词:** 功能梯度板; 光滑有限元法; 代理模型; 频率优化; 混合插值

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## Smoothed Finite Element Method for Natural Frequency Optimization of Functionally Graded Plates

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**Abstract:** Optimization of the natural frequencies of exponential functionally graded plates is a critical issue in the engineering field, playing a vital role in enhancing the dynamic performance of plate structures. In response to this challenge, this paper presents an innovative optimization approach that synergistically integrates the smoothed finite element method (SFEM) with surrogate models, aiming to address the problem with both high efficiency and precision. Based on the first order shear deformation theory, a SFEM is established for free vibration analyses of functionally graded plates. During the computation of the system stiffness matrix, gradient smoothing operations are applied to bending strains within smoothed domains, effectively improving computational accuracy. To overcome the shear locking phenomenon, different interpolation forms are adopted to treat bending strains and shear strains separately. For natural frequency optimization, a series of sample points are selected, and their corresponding natural frequencies are calculated using the SFEM. Subsequently, a surrogate model is established to map the relationship between the gradient index and the natural frequencies. The golden section method is employed to determine the optimal gradient index that achieves preset natural frequency targets. Numerical examples demonstrate that the surrogate model based on piecewise cubic Hermite interpolation exhibits high computational accuracy. Moreover, the surrogate model-based optimization significantly reduces the number of SFEM frequency calculations required, substantially enhancing optimization efficiency. This approach provides an efficient and practical method for optimizing natural frequencies of exponential functionally graded plates.

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**Key words:** functionally graded plates; smoothed finite element method; surrogate model; natural frequency optimization; mixed interpolation

作为一种新型复合材料, 功能梯度材料是一种材料宏观属性在空间位置上呈现梯度变化的非均质复合材料<sup>[1-2]</sup>. 得益于其赋予结构的高耐热性、高强度和高刚度等核心优势, 功能梯度材料被广泛应用于航空航天、核能、生物医学、机械工程及建筑等领域<sup>[3-4]</sup>. 功能梯度材料的显著优势在于其组成材料的分布具有高度的可设计性<sup>[5-7]</sup>. 在航空航天、机械制造、生物医学等不同工程领域, 可根据具体的力学性能需求, 通过调整材料分布来优化结构的力学响应, 从而充分发挥材料性能优势, 并满足多样化的工程需求.

板、壳和梁仍然是功能梯度材料结构的主要形式, 因此许多学者对功能梯度板的力学行为进行了研究. Sundararajan 等<sup>[8]</sup>采用有限元法分析了功能梯度方板和斜板的非线性动响应问题. 尹硕辉等<sup>[9]</sup>采用等几何有限元法分析了功能梯度板的自由振动问题. Zhao 等<sup>[10]</sup>、Neves 等<sup>[11]</sup>和 Chen 等<sup>[12-13]</sup>采用无网格法进行了功能梯度板的自由振动分析. 此外, 段文峰等<sup>[14]</sup>采用光滑有限元法分析了功能梯度板的自由振动问题. 为获得功能梯度板的最佳力学性能, 可在现有数值仿真和解析计算结果的基础上实施优化设计. Ashjari 等<sup>[15]</sup>以减轻功能梯度板质量为优化目标, 在满足应力与挠度约束的前提下, 优化了其厚度方向的材料梯度分布. Taheri 等<sup>[16]</sup>利用几何曲面表征组分体积分数分布, 并借助遗传算法优化, 成功减小了最大热应力值. 魏星等<sup>[17-18]</sup>采用自然单元法对功能梯度板的固有频率优化问题进行了研究.

本文采用光滑有限元法(Smoothed Finite Element Method, SFEM)和代理模型对功能梯度板固有频率进行优化. 采用梯度光滑技术<sup>[19-20]</sup>重构光滑域内应变场, 显著降低了光滑有限元法对网格畸变的敏感性. 相较于传统有限元法, 光滑有限元法不仅可以避免复杂的形函数导数计算, 而且通常可以改善求解精度<sup>[21-23]</sup>. 针对剪切闭锁问题, 引入张量分量混合插值技术<sup>[24-25]</sup>, 通过在自然坐标系下对剪切应变分量实施独立低阶插值, 有效弱化了位移场与剪切应变的耦合. 在优化过程里, 为了寻求最优解, 需对固有频率进行迭代计算. 因此, 光滑有限元法会被多次调用以完成求解工作. 本文利用分段三次 Hermite 插值建立材料梯度指数与固有频率关系的代理模型, 并应用黄金分割法<sup>[26]</sup>优化功能梯度板的梯度指数分布. 数值算例分析表明, 本文所提的功能梯度板固有频率优化的光滑有限元法具有较高的计算精度, 并且值得推广应用.

## 1 功能梯度材料属性

如图 1 所示, 厚度为  $h$  的功能梯度板从陶瓷材料连续过渡到金属材料, 且陶瓷的体积分数可以假定为

$$V_c = (1/2 + z/h)^n, \quad V_m = 1 - V_c \quad (n \geq 0) \quad (1)$$

式中,  $n$  为梯度指数,  $V_c$  和  $V_m$  分别为陶瓷和金属的体积分数. 功能梯度材料的弹性模量  $E$ , 泊松比  $\nu$  及材料密度  $\rho$  等有效力学参数  $P(z)$  可写为

$$P(z) = P_c V_c + P_m V_m \quad (2)$$

式中,  $P_c$  和  $P_m$  分别为陶瓷和金属的材料参数. 当  $n = 0$  时,  $P(z) = P_c$ ; 当  $n = \infty$  时,  $P(z) = P_m$ .

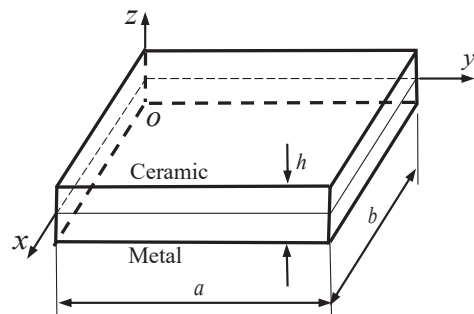


图 1 功能梯度板结构图

Fig.1 Layout of the functionally graded plate structure

## 2 自由振动分析的光滑有限元法

基于一阶剪切变形理论, 板内任意点的位移场可写为

$$u(x, y, z) = u_0(x, y) + z\beta_x(x, y) \quad (3)$$

$$v(x, y, z) = v_0(x, y) + z\beta_y(x, y) \quad (4)$$

$$w(x, y, z) = w_0(x, y) \quad (5)$$

式中,  $u_0$ ,  $v_0$  和  $w_0$  为功能梯度板中面位移,  $\beta_x$  和  $\beta_y$  为横截面转角分量. 根据式(3)~式(5), 我们有

$$\boldsymbol{\varepsilon}_m = \left[ \frac{\partial u_0}{\partial x} \quad \frac{\partial v_0}{\partial y} \quad \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} \right]^T \quad (6)$$

$$\boldsymbol{\varepsilon}_b = \left[ \frac{\partial \beta_x}{\partial x} \quad \frac{\partial \beta_y}{\partial y} \quad \frac{\partial \beta_x}{\partial y} + \frac{\partial \beta_y}{\partial x} \right]^T \quad (7)$$

$$\boldsymbol{\gamma} = \left[ \frac{\partial w_0}{\partial x} + \beta_x \quad \frac{\partial w_0}{\partial y} + \beta_y \right]^T \quad (8)$$

功能梯度板自由振动分析的平衡方程弱形式可写为

$$\int_{\Omega} \delta \boldsymbol{\varepsilon}^T \mathbf{D} \boldsymbol{\varepsilon} d\Omega + \int_{\Omega} \delta \boldsymbol{\gamma}^T \mathbf{D}^s \boldsymbol{\gamma} d\Omega + \int_{\Omega} \delta \mathbf{u}^T \bar{\mathbf{m}} \ddot{\mathbf{u}} d\Omega = 0 \quad (9)$$

式中,

$$\boldsymbol{\varepsilon} = [\boldsymbol{\varepsilon}_m^T, \boldsymbol{\kappa}^T]^T, \quad \mathbf{D} = \begin{bmatrix} \mathbf{D}^m & \bar{\mathbf{B}} \\ \bar{\mathbf{B}} & \mathbf{D}^b \end{bmatrix} \quad (10)$$

$$\bar{\mathbf{m}} = \begin{bmatrix} I_0 & 0 & 0 & I_1 & 0 \\ 0 & I_0 & 0 & 0 & I_1 \\ 0 & 0 & I_0 & 0 & 0 \\ I_1 & 0 & 0 & I_2 & 0 \\ 0 & I_1 & 0 & 0 & I_2 \end{bmatrix} \quad (11)$$

$$(\mathbf{D}^m, \bar{\mathbf{B}}, \mathbf{D}^b) = \int_{-h/2}^{h/2} (1, z, z^2) \mathbf{E}(z) dz \quad (12)$$

$$\mathbf{D}^s = \alpha \int_{-h/2}^{h/2} \mathbf{G}(z) dz \quad (13)$$

$$(I_0, I_1, I_2) = \int_{-h/2}^{h/2} (1, z, z^2) \rho(z) dz \quad (14)$$

式中,  $\alpha$  为剪切修正因子, 且有

$$\mathbf{E}(z) = \frac{E(z)}{1 - \nu(z)^2} \begin{bmatrix} 1 & \nu(z) & 0 \\ \nu(z) & 1 & 0 \\ 0 & 0 & \frac{1 - \nu(z)}{2} \end{bmatrix} \quad (15)$$

$$\mathbf{G}(z) = \frac{E(z)}{2(1 + \nu(z))} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (16)$$

将问题域  $\Omega$  离散为  $N_e$  个单元,  $\Omega = \bigcup_{i=1}^{N_e} \Omega_i^e$ ,  $\Omega_i^e \cap \Omega_j^e = \emptyset$ ,  $i \neq j$ , 在单元内增加节点形成  $n_e^s \in [1, \infty)$

个光滑域. 单元内任一点的位移场可写为

$$\mathbf{u} = [u_0 \quad v_0 \quad w_0 \quad \beta_x \quad \beta_y]^T = \sum_{i=1}^4 N_i \mathbf{q}_i \quad (17)$$

式中,  $N_i$  为形函数,  $\mathbf{q}_i = [u_{0i} \quad v_{0i} \quad w_{0i} \quad \beta_{xi} \quad \beta_{yi}]^T$  为节点  $i$  处的广义位移. 利用应变光滑技术, 光滑域内点  $\mathbf{x}_c$  处的中面面内应变可写为

$$\bar{\boldsymbol{\varepsilon}}_m(\mathbf{x}_c) = \frac{1}{A_c} \int_{\Omega_c} \boldsymbol{\varepsilon}_m d\Omega \quad (18)$$

式中,  $A_c$  为光滑域  $\Omega_c$  的面积. 将式(17)代入式(6), 再将结果代入式(18), 我们有

$$\bar{\boldsymbol{\varepsilon}}_m(\mathbf{x}_c) = \sum_{i=1}^4 \bar{\mathbf{B}}_{ci}^m \mathbf{q}_i \quad (19)$$

式中,

$$\bar{\mathbf{B}}_{ci}^m = \frac{1}{A_c} \int_{\Gamma_c} \begin{pmatrix} N_i n_x & 0 & 0 & 0 & 0 \\ 0 & N_i n_y & 0 & 0 & 0 \\ N_i n_y & N_i n_x & 0 & 0 & 0 \end{pmatrix} d\Gamma \quad (20)$$

式中,  $\Gamma_c$  为光滑域  $\Omega_c$  的边界,  $n_x$  和  $n_y$  分别为单位外法线向量在  $x$  和  $y$  方向的分量. 若光滑域  $\Omega_c$  的每条边  $\Gamma_c$  采用单点高斯积分, 则式(20)可以进一步写为

$$\bar{\mathbf{B}}_{ci}^m = \frac{1}{A_c} \sum_{b=1}^4 \begin{pmatrix} N_i(\mathbf{x}_b^G) n_x & 0 & 0 & 0 & 0 \\ 0 & N_i(\mathbf{x}_b^G) n_y & 0 & 0 & 0 \\ N_i(\mathbf{x}_b^G) n_y & N_i(\mathbf{x}_b^G) n_x & 0 & 0 & 0 \end{pmatrix} l_b^c \quad (21)$$

式中,  $\mathbf{x}_b^G$  为光滑域边界  $\Gamma_{cb}$  上的高斯积分点,  $l_b^c$  为该边界的长度.

同理, 光滑域内点  $\mathbf{x}_c$  处的曲率可以写为

$$\bar{\mathbf{k}}(\mathbf{x}_c) = \sum_{i=1}^4 \bar{\mathbf{B}}_{ci}^b \mathbf{q}_i \quad (22)$$

式中,

$$\bar{\mathbf{B}}_{ci}^b = \frac{1}{A_c} \sum_{b=1}^4 \begin{pmatrix} 0 & 0 & 0 & N_i(\mathbf{x}_b^G) n_x & 0 \\ 0 & 0 & 0 & 0 & N_i(\mathbf{x}_b^G) n_y \\ 0 & 0 & 0 & N_i(\mathbf{x}_b^G) n_y & N_i(\mathbf{x}_b^G) n_x \end{pmatrix} l_b^c \quad (23)$$

为了克服剪切自锁问题, 我们采用张量分量混合插值技术<sup>[24, 25]</sup>. 等参坐标  $(\zeta, \eta)$  下, 单元内的剪切应变  $\boldsymbol{\gamma}$  可以表达为

$$\boldsymbol{\gamma} = \begin{bmatrix} \gamma_x \\ \gamma_y \end{bmatrix} = \mathbf{J}^{-1} \begin{bmatrix} \gamma_\zeta \\ \gamma_\eta \end{bmatrix} = \mathbf{J}^{-1} \bar{\mathbf{N}} [\gamma_\eta^A \quad \gamma_\zeta^B \quad \gamma_\eta^C \quad \gamma_\zeta^D] \quad (24)$$

式中,  $\mathbf{J}$  为雅可比矩阵, 点  $A$ 、 $B$ 、 $C$  和  $D$  如图 2 所示, 且

$$\bar{\mathbf{N}} = \frac{1}{2} \begin{bmatrix} 0 & 1-\eta & 0 & 1+\eta \\ 1-\zeta & 0 & 1+\zeta & 0 \end{bmatrix} \quad (25)$$

根据式(24), 四边形单元内任一点的剪切应变  $\boldsymbol{\gamma}(\zeta, \eta)$  可表达为

$$\boldsymbol{\gamma}(\zeta, \eta) = \sum_{i=1}^4 \mathbf{B}_i^s(\zeta, \eta) \mathbf{q}_i \quad (26)$$

式中,

$$\mathbf{B}_i^s = \mathbf{J}^{-1} \begin{bmatrix} 0 & 0 & N_{i,\zeta} & \bar{N}_\zeta^i x_{,\zeta}^i & \bar{N}_\zeta^i y_{,\zeta}^i \\ 0 & 0 & N_{i,\eta} & \bar{N}_\eta^i x_{,\eta}^i & \bar{N}_\eta^i y_{,\eta}^i \end{bmatrix} \quad (27)$$

式中,  $x^i$  和  $y^i$  为四边形单元节点的整体坐标, 且

$$\bar{N}_\zeta^1 = \bar{N}_\zeta^2 = (1-\eta)/4, \quad \bar{N}_\zeta^3 = \bar{N}_\zeta^4 = (1+\eta)/4 \quad (28)$$

$$\bar{N}_\eta^1 = \bar{N}_\eta^4 = (1-\zeta)/4, \quad \bar{N}_\eta^2 = \bar{N}_\eta^3 = (1+\zeta)/4 \quad (29)$$

将式(17)、式(19)、式(22)和式(26)代入式(9), 并考虑简谐振动条件, 可得功能梯度板自由振动问题的特征方程

$$(\mathbf{K} - \omega^2 \mathbf{M})\mathbf{U} = 0 \quad (30)$$

式中,  $\mathbf{K}$  和  $\mathbf{M}$  分别为整体刚度矩阵和质量矩阵,  $\omega$  为固有频率,  $\mathbf{U}$  为所有节点的广义位移幅值, 且

$$\mathbf{K}_{ij} = \sum_{c=1}^{N_s} A_c \left( (\bar{\mathbf{B}}_{ci}^m)^T \mathbf{D}^m \bar{\mathbf{B}}_{cj}^m \right) + \sum_{c=1}^{N_s} A_c \left( (\bar{\mathbf{B}}_{ci}^b)^T \mathbf{D}^b \bar{\mathbf{B}}_{cj}^b \right) + \sum_{c=1}^{N_s} A_c \left( (\bar{\mathbf{B}}_{ci}^m)^T \bar{\mathbf{B}} \bar{\mathbf{B}}_{cj}^b + (\bar{\mathbf{B}}_{ci}^b)^T \bar{\mathbf{B}} \bar{\mathbf{B}}_{cj}^m \right) + \sum_{k=1}^{N_c} \int_{\Omega_k} (\mathbf{B}_i^s)^T \mathbf{D}^s (\mathbf{B}_j^s) d\Omega \quad (31)$$

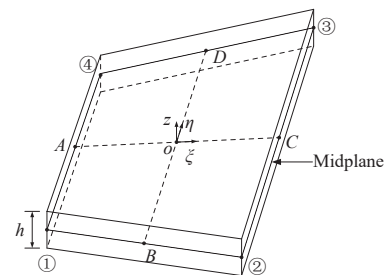


图2 功能梯度板四边形单元  
Fig.2 Quadrilateral element of functionally graded plate

$$\mathbf{M}_{ij} = \sum_{k=1}^{N_e} \int_{\Omega_k^e} N_i \bar{\mathbf{m}} N_j d\Omega \quad (32)$$

式中,  $N_s = N_e \times n_e^s$  为光滑域总数. 式(30)的非零解条件可导出特征值方程, 其解包括固有频率  $\omega$  与对应振型向量. 本文中仅需计算第一阶固有频率.

### 3 功能梯度板基频优化模型

当板的厚度与边界条件被确定时, 板的固有频率  $\omega$  直接取决于其组成材料的分布情况. 也就是说, 固有频率  $\omega$  可以看作是梯度指数  $n$  的函数. 本文选择对第一阶固有频率进行优化, 旨在使其达到预先设定的某个特定值  $\omega_{pr}$ . 该优化问题的数学表达式可以写为

$$\begin{aligned} \min f(n) &= (1 - \omega_1(n)/\omega_{pr})^2 \\ \text{s.t. } n &\geq 0 \end{aligned} \quad (33)$$

黄金分割法求解该优化问题需频繁调用光滑有限元计算功能梯度板的第一阶固有频率, 计算效率受迭代次数制约. 为此, 本文构建代理模型加速优化过程, 具体计算过程可写为: (1) 在梯度指数设计空间抽样, 并计算其对应的第一阶固有频率; (2) 基于样本点建立分段三次样条插值、分段三次 Hermite 插值或拉格朗日插值的代理模型; (3) 基于代理模型执行黄金分割优化, 进而获得最优梯度指数.

## 4 数值算例

针对 Al/Al<sub>2</sub>O<sub>3</sub> 功能梯度板, 本文采用黄金分割法对梯度指数  $n$  的优化设计, 实现不同边界条件下的第一阶固有频率目标值. 功能梯度板的材料参数为: Al 的弹性模量  $E_m = 70 \text{ GPa}$ , 泊松比  $\nu_m = 0.3$ , 质量密度  $\rho_m = 2707 \text{ kg/m}^3$ ; Al<sub>2</sub>O<sub>3</sub> 的弹性模量  $E_c = 380 \text{ GPa}$ , 泊松比  $\nu_c = 0.3$ , 质量密度  $\rho_c = 3800 \text{ kg/m}^3$ . 典型的边界条件可写为

$$\text{简支边: } \begin{cases} v_0 = w_0 = \beta_y = 0 & (x = 0, a) \\ u_0 = w_0 = \beta_x = 0 & (y = 0, b) \end{cases} \quad (34)$$

$$\text{固支边: } u_0 = v_0 = w_0 = \beta_x = \beta_y = 0 \quad (35)$$

#### 4.1 四边简支方板

针对厚跨比  $h/a = 0.2$  的四边简支方板 ( $a = b = 1$ ), 采用光滑有限元法计算梯度指数  $n$  为 0.5、1 和 4 时的一阶无量纲固有频率  $\bar{\omega} = \omega h \sqrt{\rho_c/E_c}$ , 结果分别为 0.180 8、0.163 3 和 0.139 8. 计算中采用  $24 \times 24$  四边形单元网格, 并将每个单元划分为 4 个光滑域. 为了验证代理模型的精度, 在梯度指数 [0.1, 12] 内选取 5~8 个对数等距采样点, 分别采用分段三次样条插值、分段三次 Hermite 插值和拉格朗日插值三种方法<sup>[26]</sup>构建代理模型. 随后, 在该区间内生成 50 个均匀分布测试点, 分别通过代理模型和光滑有限元法计算其第一阶无量纲固有频率. 代理模型的均方误差计算如下

$$e = \frac{1}{N} \sum_{i=1}^N (\bar{\omega}_i - \bar{\omega}_i^*)^2 \quad (36)$$

式中,  $\omega_i$  和  $\omega_i^*$  分别为采用代理模型和光滑有限元法计算得到的无量纲第一阶固有频率,  $N$  为测试点总数. 显然, 代理模型的均方误差越小, 其计算精度越高. 表 1 列出了三种插值方法构建的代理模型均方误差. 结果表明, 分段三次 Hermite 插值方法的精度最高. 因此, 本文将采用该方法构建代理模型. 表 2 给出了目标第一阶固有频率及其对应的最优梯度指数  $n$ , 并对比了直接优化方法和代理模型优化方法所

得的梯度指数结果及其调用光滑有限元法的次数. 在代理模型优化中, 由于仅需 8 个采样点构建模型, 整个优化过程仅调用光滑有限元法 8 次. 相比之下, 直接优化方法需要调用光滑有限元法 48~56 次, 计算量达到代理模型方法的 6~7 倍.

表 1 功能梯度方板固有频率代理模型的计算精度对比

Tab.1 Comparison of computational accuracy for surrogate models in predicting natural frequency of functionally graded square plate

Number of sample points	Interpolation method		
	Piecewise cubic spline	Lagrange	Piecewise cubic Hermite
5	$1.216 \times 10^{-3}$	$3.068 \times 10^{-2}$	$1.116 \times 10^{-6}$
6	$9.097 \times 10^{-5}$	$2.321 \times 10^{-2}$	$5.152 \times 10^{-7}$
7	$7.134 \times 10^{-6}$	$6.740 \times 10^{-2}$	$2.514 \times 10^{-7}$
8	$6.974 \times 10^{-7}$	3.980	$5.020 \times 10^{-8}$

表 2 四边简支功能梯度方板的优化结果

Tab.2 Optimum results for the simply supported functionally graded square plate

Meshing	Algorithm	Objective dimensionless frequency $\bar{\omega}_{pr}$					
		0.180 8 ( $n=0.5$ )		0.163 3 ( $n=1$ )		0.139 8 ( $n=4$ )	
		Result	Number of SFEM evaluation	Result	Number of SFEM evaluation	Result	Number of SFEM evaluation
16 × 16	Direct	0.510 2	50	1.016 6	52	4.190 1	50
	Surrogate	0.511 4	8	1.025 5	8	4.194 0	8
18 × 18	Direct	0.507 3	50	1.011 8	50	4.136 2	50
	Surrogate	0.508 5	8	1.020 8	8	4.141 5	8
20 × 20	Direct	0.505 2	49	1.008 4	50	4.098 4	50
	Surrogate	0.506 5	8	1.017 5	8	4.104 7	8
22 × 22	Direct	0.503 7	48	1.005 9	50	4.070 8	50
	Surrogate	0.505 0	8	1.015 0	8	4.077 8	8
24 × 24	Direct	0.502 5	50	1.004 0	50	4.049 9	50
	Surrogate	0.503 9	8	1.013 0	8	4.057 5	8

## 4.2 固支圆板

考虑半径  $R = 1.0$  的固支圆板, 计算中采用的网格划分见图 3. 文献[27]给出梯度指数  $n = 1$  时两组厚跨比下的无量纲第一阶固有频率: 当  $h/R = 0.01$  时  $\bar{\omega} = 100\omega h \sqrt{\rho_c/E_c} = 0.023 6$ , 当  $h/R = 0.2$  时  $\bar{\omega} = 8.653 5$ . 表 3 对比了直接优化与代理模型优化的梯度指数结果和光滑有限元法的调用次数, 表明代理模型优化方法能在精度损失可控的前提下显著降低计算成本. 特别地, 薄板工况( $h/R = 0.01$ )的优化结果仍保持较高精度, 验证了光滑有限元法克服剪切自锁的有效性.

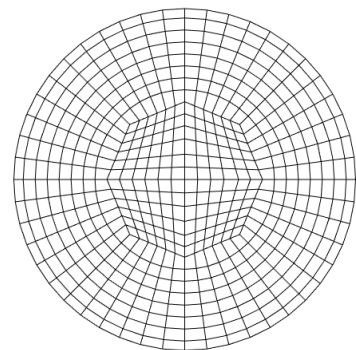


图 3 四节点四边形单元分布

Fig.3 Distribution of Q4 elements

表3 固支功能梯度圆板的优化结果  
Tab.3 Optimum results for the clamped functionally graded circular plate

$h/R$	$\bar{\omega}_{pr}$	Algorithm	Result	Number of SFEM evaluation
0.01	0.023 6 ( $n=1$ )	Direct	1.017 4	54
		Surrogate	1.027 1	8
0.2	8.653 5 ( $n=1$ )	Direct	1.022 5	50
		Surrogate	1.031 4	8

## 5 结论

本文首次提出了光滑有限元法与代理模型相结合进行功能梯度板固有频率优化的方法, 进一步拓展了光滑有限元法的应用范围. 根据本文研究分析和算例求解结果可以看出

(1) 光滑有限元法通过梯度光滑技术重构非局部梯度场以软化系统刚度矩阵, 不仅避免了形函数导数的计算, 而且具有网格适应性高及计算精度优势, 在求解功能梯度板的自由振动问题时仍然具有较高的计算精度.

(2) 张量分量混合插值技术通过在单元边中点独立插值剪切应变分量, 弱化位移场与剪切应变的耦合, 消除低阶插值导致的虚假剪切刚度, 有效克服剪切自锁.

(3) 采用拉格朗日、分段三次样条和分段三次 Hermite 插值法构建功能梯度板代理模型, 并通过计算测试点处的均方误差比较精度, 结果显示分段三次 Hermite 插值法最优.

(4) 相对于直接优化方法, 基于代理模型优化得到的功能梯度指数结果相近, 能在保证一定精度的同时, 大幅降低计算工作量, 提升优化效率.

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