双曲型方程的 Crank–Nicolson 块中心差分方法

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摘要 用 Crank–Nicolson 块中心差分法研究了有界区域上的线性双曲型微分方程的数值解,此方法以块中心差分方法和抛物型的 Crank–Nicolson 格式为基础。在非等距剖分的网格上得到了近似解和解的一阶导数。其特点是近似解按离散的 L^2 模达到最优阶误差估计,解的一阶导数的近似解达到超收敛误差估计,达到和近似解同样的精度。本文所讨论的方法,在计算量上没有增加。数值试验结果与理论分析一致,说明格式具有高效的收敛性。

关键词 双曲型微分方程;Crank–Nicolson 块中心差分方法;误差估计

中图分类号 O241.82 **文献标识码** A **doi** 10.3981/j.issn.1000–7857.2011.09.009

Crank–Nicolson Block–centered Finite Differences Method for Hyperbolic Problems

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Abstract The Crank–Nicolson block–centered finite difference method studies the solution of the linear hyperbolic differential problems in the bounded domain with sufficiently smooth data. This method is based on both block–center finite difference method and parabolic Crank–Nicolson format. Both the approximate solution and its first derivatives are obtained for all non–uniform grids. Its characteristics are that the approximate solution according to the discrete L^2–norm is achieved optimal order error estimation, and the approximate solution of the first derivatives is reached at super convergence error estimation. This method does not increase the calculation. Numerical tests are identical with theoretical analysis; it explains that the format possesses the efficient convergence.

Keywords hyperbolic differential equation; Crank–Nicolson block–centered finite differences method; error estimation

0 引言

双曲型方程在物理、工程领域里有着广泛的应用,受到许多数学和工程技术方面专家的关注和重视。不管从理论还是数值方面,都有必要对它进行全面深入的研究。针对双曲型方程的数值计算已经有了许多差分格式和有限元方法[1–4]。这些差分格式是在等距剖分网格上建立的,对一些分布不均匀的问题求得的近似解与真解相差较大。而对一些在近似解要求不太严格的问题有限元方法比较繁琐。1991 年王申林[5]讨论了解拟线性双曲型积分微分方程的块中心差分方法,其共同特点为近似解按离散的 L^2 模达到最优阶误差估计,解的一阶导数的近似解达到超收敛误差估计。Weiser 和 Wheeler[5]对线性椭圆形和抛物型方程提出了块中心差分方法。本文将其方法推广到了双曲型方程。该方法的优越性在于不仅可以求出近似解还可以求出近似解的导数,而且解的一阶导数和解具有同样的精度。

考虑如下线性双曲型微分方程的初边值问题:

$$\frac{\partial^2 p}{\partial t^2}-\nabla\cdot\boldsymbol{a}(x,y)\nabla p=f(x,y,t)\quad (x,y,t)\in\Omega\times J \tag{1}$$

$$\nabla p\cdot\boldsymbol{n}_{\Omega}=0\quad (x,y,t)\in\partial\Omega\times J \tag{2}$$

$$p(x,y,0)=p^0(x,y)\quad \frac{\partial p(x,y,0)}{\partial t}=q^0(x,y)\quad (x,y)\in\Omega \tag{3}$$

其中,$\boldsymbol{a}=(a^x,a^y)$,$J=[0,T]$,f,p^0,q^0 为已知函数,p 为所求函数。

收稿日期:2010–09–29;修回日期:2011–03–08

基金项目:国家自然科学基金项目(10671057)

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Ω 为二维有界区域,且有光滑边界 $\partial\Omega,\boldsymbol{n}_\Omega$ 是 $\partial\Omega$ 的外法向量。由文献[6],可设 Ω 为矩形区域,不妨设 $\Omega=(0,1)\times(0,1)$。

对任意的 $(x,y,t)\in\Omega\times J$,对上述问题做如下假设:

(i) 存在正常数 a_0^x,a_1^x,a_0^y,a_1^y,使得

$$0<a_0^x\leqslant a^x\leqslant a_1^x,0<a_0^y\leqslant a^y\leqslant a_1^y \quad c_0=\min\{a_0^x,a_0^y\}$$

(ii) $\boldsymbol{a},p^0,q^0,f$ 均具有适当的光滑性。令

$$\boldsymbol{u}=(u^x,u^y)=\left(-\frac{\partial p}{\partial x},-\frac{\partial p}{\partial y}\right)$$

则式(1)、式(2)等价于式(4)和式(5)。

$$\frac{\partial^2 p}{\partial t^2}+\frac{\partial}{\partial x}(a^x u^x)+\frac{\partial}{\partial y}(a^y u^y)=f(x,y,t) \quad (x,y,t)\in\Omega\times(0,T] \tag{4}$$

$$\boldsymbol{u}=0 \quad (x,y,t)\in\partial\Omega\times[0,T] \tag{5}$$

1 网格函数空间

区域 Ω 的剖分 $\delta_x\times\delta_y$ 定义为

$$\delta_x:\ 0=x_{\frac{1}{2}}<x_{\frac{3}{2}}<x_{\frac{5}{2}}<\cdots<x_{N_x-\frac{1}{2}}<x_{N_x+\frac{1}{2}}=1$$

$$\delta_y:\ 0=y_{\frac{1}{2}}<y_{\frac{3}{2}}<y_{\frac{5}{2}}<\cdots<y_{N_y-\frac{1}{2}}<y_{N_y+\frac{1}{2}}=1$$

式中,h_i 和 k_j 分别为水平和垂直方向的步长,$i=1,2,\cdots,N_x,j=1,2,\cdots,N_y$,其中 N_x,N_y 为整数。

定义

$$x_i=\frac{x_{i+\frac{1}{2}}+x_{i-\frac{1}{2}}}{2} \qquad y_j=\frac{y_{j+\frac{1}{2}}+y_{j-\frac{1}{2}}}{2}$$

$$h_i=x_{i+\frac{1}{2}}-x_{i-\frac{1}{2}} \qquad k_j=y_{j+\frac{1}{2}}-y_{j-\frac{1}{2}}$$

$$h_{i+\frac{1}{2}}=x_{i+1}-x_i=\frac{h_i+h_{i+1}}{2} \qquad k_{j+\frac{1}{2}}=y_{j+1}-y_j=\frac{k_j+k_{j+1}}{2}$$

$$h=\max_i\{h_i\} \qquad k=\max_j\{k_j\}$$

$$\Omega_i=\left(x_{i-\frac{1}{2}},x_{i+\frac{1}{2}}\right) \qquad \Omega_{i,j}=\left(x_{i-\frac{1}{2}},x_{i+\frac{1}{2}}\right)\times\left(y_{j-\frac{1}{2}},y_{j+\frac{1}{2}}\right)$$

记

$$g_{i,j}=g(x_i,y_j) \qquad g_{i+\frac{1}{2},j}=g\left(x_{i+\frac{1}{2}},y_j\right) \qquad g_{i,j+\frac{1}{2}}=g\left(x_i,y_{j+\frac{1}{2}}\right)$$

$$[d_x g]_{i+\frac{1}{2},j}=\frac{g_{i+1,j}-g_{i,j}}{h_{i+\frac{1}{2}}} \qquad [d_y g]_{i,j+\frac{1}{2}}=\frac{g_{i,j+1}-g_{i,j}}{k_{j+\frac{1}{2}}}$$

$$[D_x g]_{i,j}=\frac{g_{i+\frac{1}{2},j}-g_{i-\frac{1}{2},j}}{h_i} \qquad [D_y g]_{i,j}=\frac{g_{i,j+\frac{1}{2}}-g_{i,j-\frac{1}{2}}}{k_j}$$

设 τ 为时间步长,$t_n=n\tau,n=0,1,2,\cdots,N,N=\left[\frac{T}{\tau}\right]$。记

$$g_{i,j}^n=g(x_i,y_j,t^n) \qquad \delta_t^2 g^n=\frac{1}{\tau^2}(g^{n+1}-2g^n+g^{n-1})$$

$$d_t g^n=\frac{1}{\tau}(g^n-g^{n-1}) \qquad \Delta g^n=g^n-g^{n-2}$$

用 $S,S^{(1)},S^{(2)}$ 表示如下 3 类网格函数族:

$$S=\{f_{i,j}\,|\,i=1,2,\cdots,N_x;j=1,2,\cdots,N_y\}$$

$$S^{(1)}=\left\{f_{i-\frac{1}{2},j}\,\middle|\,f_{\frac{1}{2},j}=f_{N_x+\frac{1}{2},j}=0,i=1,2,\cdots,N_x+1;j=1,2,\cdots,N_y\right\}$$

$$S^{(2)}=\left\{f_{i,j-\frac{1}{2}}\,\middle|\,f_{i,\frac{1}{2}}=f_{i,N_y+\frac{1}{2}}=0,i=1,2,\cdots,N_x;j=1,2,\cdots,N_y+1\right\}$$

对 3 类网格函数族分别定义内积和范数:

$$(f,g)_M=\sum_{i=1}^{N_x}\sum_{j=1}^{N_y}h_ik_jf_{i,j}\cdot g_{i,j} \qquad \|f\|_M^2=(f,f)_M$$

$$(f,g)_x=\sum_{i=2}^{N_x}\sum_{j=1}^{N_y}h_{i-\frac{1}{2}}k_jf_{i-\frac{1}{2},j}\cdot g_{i-\frac{1}{2},j} \qquad \|f\|_x^2=(f,f)_x$$

$$(f,g)_y=\sum_{i=1}^{N_x}\sum_{j=2}^{N_y}h_ik_{j-\frac{1}{2}}f_{i,j-\frac{1}{2}}\cdot g_{i,j-\frac{1}{2}} \qquad \|f\|_y^2=(f,f)_y$$

记

$$W_q^b=\left\{f(x,y):\text{当 } m+e\leqslant b,q<+\infty \text{ 时},\left(\left|\frac{\partial^{m+e}f}{\partial x^m\partial y^e}\right|^q,1\right)\text{有界;}\right.$$

$$\left.\text{当}q=\infty\text{ 时},\left|\frac{\partial^{m+e}f}{\partial x^m\partial y^e}\right|^q\text{有界}\right\}$$

定义 1[6] 设 $f=(f^x,f^y)\in S^{(1)}\times S^{(2)}$,定义范数 $\|\,|f|\,\|$ 为

$$\|\,|f|\,\|^2=(f^x,f^x)_x+(f^y,f^y)_y$$

定义 2 设 $\psi\in c(\bar{\Omega})$,定义

$$\|\psi\|_{l_x^2}^2=\sum_{i=1}^{N_x}\sum_{j=1}^{N_y}h_ik_j\left\{\max|\psi(x,y_j)|^2:|x-x_i|\leqslant\frac{h_i}{2}\right\}$$

$$\|\psi\|_{l_y^2}^2=\sum_{i=1}^{N_x}\sum_{j=1}^{N_y}h_ik_j\left\{\max|\psi(x_i,y)|^2:|y-y_j|\leqslant\frac{k_j}{2}\right\}$$

引理 1[6] 设 $(f,g^x,g^y)\in S\times S^{(1)}\times S^{(2)}$,则有

$$(f,D_xg^x)_M=-(d_xf,g^x)_x \qquad (f,D_yg^y)_M=-(d_yf,g^y)_y$$

引理 2[6] 假设下列条件成立:

(1) $p\in W_3^4(\Omega)$;

(2) $\frac{\partial^{s+2}p}{\partial x^2\partial t^s},\frac{\partial^{s+2}p}{\partial y^2\partial t^s},\frac{\partial^{s+3}p}{\partial x^2\partial y\partial t^s},\frac{\partial^{s+3}p}{\partial x\partial y^2\partial t^s}$ 在 Ω 中 Lipschitz连续,$s=0,1$,

则存在函数:$\frac{\partial P}{\partial t^s}\in S,\frac{\partial U^{x,n}}{\partial t^s}\in S^{(1)},\frac{\partial U^{y,n}}{\partial t^s}\in S^{(2)}$,使得下式成立:

$$U_{i-\frac{1}{2},j}^x=[-d_xP]_{i-\frac{1}{2},j} \qquad U_{i,j-\frac{1}{2}}^y=[-d_yP]_{i,j-\frac{1}{2}}$$

$$\left\|\left|\frac{\partial^s}{\partial t^s}(p_{i,j}-P_{i,j})\right|\right\|\leqslant C(h^2+k^2)$$

$$\left\|\left|\frac{\partial^s}{\partial t^s}(u_{i-\frac{1}{2},j}^x-U_{i-\frac{1}{2},j}^x)\right|\right\|\leqslant C(h^2+k^2)$$

$$\left\|\left|\frac{\partial^s}{\partial t^s}(u_{i,j-\frac{1}{2}}^y-U_{i,j-\frac{1}{2}}^y)\right|\right\|\leqslant C(h^2+k^2)$$

其中,$C>0$ 在不同的等式中代表不同的常数。

引理 3[7] 设 $f\in S$,则有

$$\|f\|_M\leqslant\|d_xf\|_x \qquad \|f\|_M\leqslant\|d_yf\|_y$$

引理 4 设 $\{f^n\},\{g^n\}\in S$,则有

$$\left(f^n,\frac{1}{\tau}\Delta g^n\right)_M=\frac{1}{\tau}\Delta(f^n,g^n)_M-\left(\frac{1}{\tau}\Delta f^n,g^{n-2}\right)_M$$

$$\left(f^{n-2},\frac{1}{\tau}\Delta g^n\right)_M=\frac{1}{\tau}\Delta(f^n,g^n)_M-\left(\frac{1}{\tau}\Delta f^n,g^n\right)_M$$

证明

$$\left(f^n,\frac{1}{\tau}\Delta g^n\right)_M=\frac{1}{\tau}(f^n,g^n-g^{n-2})_M$$
$$=\frac{1}{\tau}[(f^n,g^n)_M-(f^{n-2},g^{n-2})_M+(f^{n-2},g^{n-2})_M-(f^n,g^{n-2})_M]$$
$$=\frac{1}{\tau}\Delta(f^n,g^n)_M-\left(\frac{1}{\tau}\Delta f^n,g^{n-2}\right)_M$$

同理可证,$\left(f^{n-2},\frac{1}{\tau}\Delta g^n\right)_M=\frac{1}{\tau}\Delta(f^n,g^n)_M-\left(\frac{1}{\tau}\Delta f^n,g^n\right)_M$ 成立。

2 块中心差分格式及误差估计

解问题(1)—(3)的块中心差分格式为:求$(P^n,U^{x,n},U^{y,n})\in S\times S^{(1)}\times S^{(2)}$,使得

$$\delta_t^2P^n_{i,j}+\left[D_x\frac{a^x(U^{x,n+1}+U^{x,n-1})}{2}\right]_{i,j}+\left[D_y\frac{a^y(U^{y,n+1}+U^{y,n-1})}{2}\right]_{i,j}=f^n_{i,j}\qquad(6)$$

$$U^{x,n}_{i+\frac{1}{2},j}=-[d_xP]^n_{i+\frac{1}{2},j}\qquad U^{y,n}_{i,j+\frac{1}{2}}=-[d_yP]^n_{i,j+\frac{1}{2}}\qquad(7)$$

$$P^n_{i,j}=p^n_{i,j}\qquad P^1_{i,j}=p^0_{i,j}+q^0_{i,j}\tau\qquad(8)$$

其中,$i=1,2,\cdots,N_x;j=1,2,\cdots,N_y;n=1,2,\cdots,N-1$。

将式(7),式(8)代入式(6)可得未知量$P^n_{i,j}$的线性代数方程组,可知系数矩阵是不可约对角占优的,因此解P^n存在且唯一。

将式(4)在(x_i,y_j,t^n)处写成下列形式:

$$\sigma_t^2p^n_{i,j}+\left[D_x\frac{a^x(u^{x,n+1}+u^{x,n-1})}{2}\right]_{i,j}+\left[D_y\frac{a^y(u^{y,n+1}+u^{y,n-1})}{2}\right]_{i,j}=f^n_{i,j}+A^n_{i,j}+B^{x,n}_{i,j}+B^{y,n}_{i,j}\qquad(9)$$

其中,

$$A^n_{i,j}=\delta_t^2p^n_{i,j}-\left(\frac{\partial^2p^n}{\partial t^2}\right)_{i,j}$$
$$B^{x,n}_{i,j}=\left[D_x\frac{a^x(u^{x,n+1}+u^{x,n-1})}{2}\right]_{i,j}-\frac{\partial}{\partial x}(a^xu^{x,n})_{i,j}$$
$$B^{y,n}_{i,j}=\left[D_y\frac{a^y(u^{y,n+1}+u^{y,n-1})}{2}\right]_{i,j}-\frac{\partial}{\partial y}(a^yu^{y,n})_{i,j}$$

可以验证下列不等式成立:

$$\|A^n\|_M^2\leqslant C\left\|\frac{1}{\tau^2}\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}}(\tau-s)^3\frac{\partial^4p}{\partial t^4}ds\right\|_M^2\leqslant C\tau^3\left\|\frac{\partial^4p}{\partial t^4}\right\|^2_{L^2(t^{n-1},t^n;M)}$$

由 Schwarz 不等式,可得

$$(B^{x,n}_{i,j})^2=\left\{\frac{\partial}{\partial x}(a^xu^{x,n})_{i,j}-[D_x(a^xu^{x,n})]_{i,j}+O(\tau^2)\right\}^2$$
$$\leqslant\frac{2}{4h^2}\left[\int_{-\frac{h_i}{2}}^{\frac{h_i}{2}}\left(\frac{h_i}{2}-|z|\right)^2\frac{\partial^3(a^xu^{x,n})}{\partial z^3}(x_i+z,y_j,t^n)dz\right]^2+O(\tau^4)$$
$$\leqslant\frac{1}{2h_i^2}\left\{\max\left\{\frac{\partial^3(a^xu^{x,n})}{\partial x^3}:|x-x_i|\leqslant\frac{h_i}{2}\right\}\right\}^2\cdot\left[\int_{-\frac{h_i}{2}}^{\frac{h_i}{2}}\left(\frac{h_i}{2}-|z|\right)^2dz\right]^2+O(\tau^4)$$
$$\leqslant\frac{h^4}{2}\left\{\max\left\{\frac{\partial^3(a^xu^{x,n})}{\partial x^3}:|x-x_i|\leqslant\frac{h_i}{2}\right\}\right\}^2+O(\tau^4)$$

故

$$\|B^{x,n}\|_M^2\leqslant\frac{h^4}{2}\left\|\frac{\partial^3(a^xu^{x,n})}{\partial x^3}\right\|^2_{l_x^2}+O(\tau^4)$$

同理,

$$\|B^{y,n}\|_M^2\leqslant\frac{k^4}{2}\left\|\frac{\partial^3(a^yu^{y,n})}{\partial y^3}\right\|^2_{l_y^2}+O(\tau^4)$$

由引理 2 可知,存在$Q_{i,j}\in S,V^x_{i+\frac{1}{2},j}\in S^{(1)},V^y_{i,j+\frac{1}{2}}\in S^{(2)}$,使得

$$V^x_{i+\frac{1}{2},j}=-(d_xQ)_{i+\frac{1}{2},j}\qquad V^y_{i,j+\frac{1}{2}}=-(d_yQ)_{i,j+\frac{1}{2}}$$

令$\xi^x=U^x-V^x,\xi^y=U^y-V^y,\beta^x=V^x-u^x,\beta^y=V^y-u^y,\alpha=Q-p,\eta=P-Q$,则有

$$\xi^x_{i+\frac{1}{2},j}=-[d_x\eta]_{i+\frac{1}{2},j}\qquad\xi^y_{i,j+\frac{1}{2}}=-[d_y\eta]_{i,j+\frac{1}{2}}$$

由式(6)—式(9)可得

$$\delta_t^2\eta^n_{i,j}+\left[D_x\frac{a^x(\xi^{x,n+1}+\xi^{x,n-1})}{2}\right]_{i,j}+\left[D_y\frac{a^y(\xi^{y,n+1}+\xi^{y,n-1})}{2}\right]_{i,j}=-A^n_{i,j}-B^{x,n}_{i,j}-B^{y,n}_{i,j}-\delta_t^2\alpha^n_{i,j}-\frac{R^{n+1}_{i,j}+R^{n-1}_{i,j}}{2}\qquad(10)$$

这里,$R^{n+1}_{i,j}=[D_xa^x\beta^{x,n+1}]_{i,j}-[D_ya^y\beta^{y,n+1}]_{i,j}$,而

$$|\delta_t^2\alpha^n_{i,j}|=\left|\frac{1}{\tau^2}\int_{-\tau}^{\tau}|\tau-s|\frac{\partial^2\alpha_{i,j}}{\partial s^2}ds\right|\leqslant C\tau^{-\frac{1}{2}}\|\alpha_{tt}\|_{L^2(t^{n-1},t^n)}$$

将式(10)与$d_t\eta^{n+1}+d_t\eta^n$作 M 内积,可得

$$(\delta_t^2\eta^n,d_t\eta^{n+1}+d_t\eta^n)_M+\left(D_x\frac{a^x(\xi^{x,n+1}+\xi^{x,n-1})}{2},d_t\eta^{n+1}+d_t\eta^n\right)_M+\left(D_y\frac{a^y(\xi^{y,n+1}+\xi^{y,n-1})}{2},d_t\eta^{n+1}+d_t\eta^n\right)_M$$
$$=(-A^n,d_t\eta^{n+1}+d_t\eta^n)_M-(B^{x,n},d_t\eta^{n+1}+d_t\eta^n)_M-(B^{y,n},d_t\eta^{n+1}+d_t\eta^n)_M-(\delta_t^2\alpha^n,d_t\eta^{n+1}+d_t\eta^n)_M-\left(\frac{R^{n+1}+R^{n-1}}{2},d_t\eta^{n+1}+d_t\eta^n\right)_M\qquad(11)$$

对式(11)逐项估计。由引理 1 可得

$$\left[D_x\frac{a^x(\xi^{x,n+1}+\xi^{x,n-1})}{2},d_t\eta^{n+1}+d_t\eta^n\right]_M+\left[D_y\frac{a^y(\xi^{y,n+1}+\xi^{y,n-1})}{2},d_t\eta^{n+1}+d_t\eta^n\right]_M$$
$$=\frac{1}{2\tau}[(a^x\xi^{x,n+1},\xi^{x,n+1})_x-(a^x\xi^{x,n-1},\xi^{x,n-1})_x+(a^y\xi^{y,n+1},\xi^{y,n+1})_y-(a^y\xi^{y,n+1},\xi^{y,n+1})_y]\qquad(12)$$

$$(\delta_t^2\eta^n,d_t\eta^{n+1}+d_t\eta^n)_M=\frac{1}{\tau}(\|d_t\eta^{n+1}\|_M^2-\|d_t\eta^{n-1}\|_M^2)\qquad(13)$$

$$|(B^{x,n},d_t\eta^{n+1}+d_t\eta^n)_M|\leqslant h^4\left\|\frac{\partial^3}{\partial x^3}(a^xu^{x,n})\right\|^2_{l_x^2}+O(\tau^4)+\frac{1}{16}\|d_t\eta^{n+1}\|_M^2+\frac{1}{16}\|d_t\eta^n\|_M^2\qquad(14)$$

$$|(B^{y,n},d_t\eta^{n+1}+d_t\eta^n)_M|\leqslant k^4\left\|\frac{\partial^3}{\partial y^3}(a^yu^{y,n})\right\|^2_{l_y^2}+O(\tau^4)+$$

$$\frac{1}{16}\|d_t\eta^{n+1}\|_M^2+\frac{1}{16}\|d_t\eta^n\|_M^2 \tag{15}$$

$$\begin{aligned}&|(\delta_t^2\alpha^n,d_t\eta^{n+1}+d_t\eta^n)_M|\leqslant C\tau^{-1}\|\alpha_{tt}\|^2_{L^2(t^{n-1},t^n;M)}+\\&\frac{1}{16}\|d_t\eta^{n+1}\|_M^2+\frac{1}{16}\|d_t\eta^n\|_M^2\end{aligned} \tag{16}$$

$$\begin{aligned}&|(A^n,d_t\eta^{n+1}+d_t\eta^n)_M|\leqslant C\tau^3\left\|\frac{\partial^4 p}{\partial t^4}\right\|^2_{L^2(t^{n-1},t^n;M)}+\\&\frac{1}{16}\|d_t\eta^{n+1}\|_M^2+\frac{1}{16}\|d_t\eta^n\|_M^2\end{aligned} \tag{17}$$

将式(12)—式(17)代入式(11)可得

$$\begin{aligned}&\frac{1}{2\tau}[(a^x\xi^{x,n+1},\xi^{x,n+1})_x-(a^x\xi^{x,n-1},\xi^{x,n-1})_x+(a^y\xi^{y,n+1},\xi^{y,n+1})_y-\\&(a^y\xi^{y,n+1},\xi^{y,n+1})_y]+\frac{1}{\tau}(\|d_t\eta^{n+1}\|_M^2-\|d_t\eta^{n-1}\|_M^2)\\&\leqslant C\tau^3\left\|\frac{\partial^4 p}{\partial t^4}\right\|^2_{L^2(t^{n-1},t^n;M)}+h^4\left\|\frac{\partial^3}{\partial x^3}(a^x u^{x,n})\right\|^2_{l_x^2}+\\&k^4\left\|\frac{\partial^3}{\partial y^3}(a^y u^{y,n})\right\|^2_{l_y^2}+O(\tau^4)+\frac{1}{4}\|d_t\eta^{n+1}\|_M^2+\frac{1}{4}\|d_t\eta^n\|_M^2+\\&C\tau^{-1}\|\alpha_{tt}\|^2_{L^2(t^{n-1},t^n;M)}+\left(\frac{R^{n+1}+R^{n-1}}{2},d_t\eta^{n+1}+d_t\eta^n\right)_M\end{aligned} \tag{18}$$

将式(18)×2τ,并对所有的 $1\leqslant n\leqslant N-1$ 求和,有

$$\begin{aligned}&(a^x\xi^{x,N},\xi^{x,N})_x+(a^x\xi^{x,N-1},\xi^{x,N-1})_x+(a^y\xi^{y,N},\xi^{y,N})_y+(a^y\xi^{y,N-1},\xi^{y,N-1})_y-\\&(a^x\xi^{x,1},\xi^{x,1})_x-(a^x\xi^{x,0},\xi^{x,0})_x-(a^y\xi^{y,1},\xi^{y,1})_y-(a^y\xi^{y,0},\xi^{y,0})_y+\\&2(\|d_t\eta^N\|_M^2+\|d_t\eta^{N-1}\|_M^2-\|d_t\eta^1\|_M^2-\|d_t\eta^0\|_M^2)\leqslant\\&C(\tau^4+h^4+k^4)+\frac{1}{2}\tau\sum_{n=1}^{N-1}\|d_t\eta^{n+1}\|_M^2+\\&\frac{1}{2}\tau\sum_{n=1}^{N-1}\|d_t\eta^n\|_M^2+C\|\alpha_{tt}\|^2_{L^2(0,T;M)}+\\&\tau\left|\sum_{n=1}^{N-1}(R^{n+1}+R^{n-1},d_t\eta^{n+1}+d_t\eta^n)_M\right|\end{aligned} \tag{19}$$

根据离散 Gronwall 不等式和引理 2,可得

$$\begin{aligned}&(a^x\xi^{x,N},\xi^{x,N})_x+(a^x\xi^{x,N-1},\xi^{x,N-1})_x+(a^y\xi^{y,N},\xi^{y,N})_y+(a^y\xi^{y,N-1},\xi^{y,N-1})_y+\\&\|d_t\eta^N\|_M^2+2\|d_t\eta^{N-1}\|_M^2\leqslant(a^x\xi^{x,1},\xi^{x,1})_x+(a^x\xi^{x,0},\xi^{x,0})_x+\\&(a^y\xi^{y,1},\xi^{y,1})_y+(a^y\xi^{y,0},\xi^{y,0})_y+2\|d_t\eta^1\|_M^2+2\|d_t\eta^0\|_M^2+\\&C(\tau^4+h^4+k^4)+\tau\left|\sum_{n=1}^{N-1}(R^{n+1}+R^{n-1},d_t\eta^{n+1}+d_t\eta^n)_M\right|\end{aligned} \tag{20}$$

由初值的选取和引理 2 可知,

$$\|\eta^0\|\leqslant C(h^2+k^2+\tau^2)\qquad \|\eta^1\|\leqslant C(h^2+k^2+\tau^2)$$

$$\||\xi^0|\|\leqslant C(h^2+k^2+\tau^2)\qquad \||\xi^1|\|\leqslant C(h^2+k^2+\tau^2)$$

故有

$$\begin{aligned}&\|d_t\eta^N\|_M^2+2\|d_t\eta^{N-1}\|_M^2+c_0\||\xi^N|\|^2+c_0\||\xi^{N-1}|\|^2\leqslant\\&C(\tau^4+h^4+k^4)+\tau\left|\sum_{n=1}^{N-1}(R^{n+1}+R^{n-1},d_t\eta^{n+1}+d_t\eta^n)_M\right|\end{aligned} \tag{21}$$

现在估计 $\tau\left|\sum_{n=1}^{N-1}(R^{n+1}+R^{n-1},d_t\eta^{n+1}+d_t\eta^n)_M\right|$。

利用引理 1 可得

$$(R^n,\eta^n)_M\leqslant\varepsilon\||\xi^n|\|^2+C\||\beta^n|\|^2$$

利用引理 4 可得

$$\begin{aligned}&\tau\left|\sum_{n=1}^{N-1}\left(R^{n+1},\frac{1}{\tau}(\eta^{n+1}-\eta^{n-1})\right)_M\right|=\tau\left|\sum_{n=1}^{N-1}\left(R^{n+1},\frac{1}{\tau}\Delta\eta^{n+1}\right)_M\right|\\&=\left|\tau\sum_{n=1}^{N-1}\left[\frac{1}{\tau}\Delta(R^{n+1},\eta^{n+1})_M-\left(\frac{1}{\tau}\Delta R^{n+1},\eta^{n-1}\right)_M\right]\right|\\&=\Big|(R^N,\eta^N)_M+(R^{N-1},\eta^{N-1})_M-(R^1,\eta^1)_M-(R^0,\eta^0)_M-\\&\tau\sum_{n=1}^{N-1}\left[\left(D_x\frac{1}{\tau}\Delta(a^x\beta^{x,n+1}),\eta^{n-1}\right)_M+\left(D_y\frac{1}{\tau}\Delta(a^y\beta^{y,n+1}),\eta^{n-1}\right)_M\right]\Big|\\&=\Big|(R^N,\eta^N)_M+(R^{N-1},\eta^{N-1})_M-(R^1,\eta^1)_M-(R^0,\eta^0)_M-\\&\tau\sum_{n=1}^{N-1}\left[\left(\frac{1}{\tau}\Delta(a^x\beta^{x,n+1}),\xi^{x,n-1}\right)_x+\left(\frac{1}{\tau}\Delta(a^y\beta^{y,n+1}),\xi^{y,n-1}\right)_y\right]\Big|\leqslant\\&\varepsilon(\||\xi^N|\|^2+\||\xi^{N-1}|\|^2+\||\xi^1|\|^2+\||\xi^0|\|^2)+\\&C(\||\beta^N|\|^2+\||\beta^{N-1}|\|^2+\||\beta^1|\|^2+\||\beta^0|\|^2)+\\&\tau\sum_{n=1}^{N-1}\left[\varepsilon\||\xi^{n-1}|\|^2+C\left(\left\|\frac{\partial\beta^x(t_n^1)}{\partial t}\right\|_x^2+\left\|\frac{\partial\beta^y(t_n^2)}{\partial t}\right\|_y^2\right)\right]\end{aligned} \tag{22}$$

这里,$t^{n-1}\leqslant t_n^1,t_n^2\leqslant t^{n+1}$。

同理可得,

$$\begin{aligned}&\tau\left|\sum_{n=1}^{N-1}\left(R^{n-1},\frac{1}{\tau}(\eta^{n+1}-\eta^{n-1})\right)_M\right|\leqslant\varepsilon(\||\xi^N|\|^2+\||\xi^{N-1}|\|^2+\\&\||\xi^1|\|^2+\||\xi^0|\|^2)+C(\||\beta^N|\|^2+\||\beta^{N-1}|\|^2+\\&\||\beta^1|\|^2+\||\beta^0|\|^2)+\tau\sum_{n=1}^{N-1}\Big[\varepsilon\||\xi^{n+1}|\|^2+\\&C\left(\left\|\frac{\partial\beta^x(t_n^3)}{\partial t}\right\|_x^2+\left\|\frac{\partial\beta^y(t_n^4)}{\partial t}\right\|_y^2\right)\Big]\end{aligned} \tag{23}$$

其中,$t^{n-1}\leqslant t_n^3,t_n^4\leqslant t^{n+1}$。

由初值的选取和引理 2 可知,

$$\begin{aligned}&\tau\left|\sum_{n=1}^{N-1}(R^{n+1}+R^{n-1},d_t\eta^{n+1}+d_t\eta^n)_M\right|\leqslant2\varepsilon(\||\xi^N|\|^2+\\&\||\xi^{N-1}|\|^2)+C(h^4)+\tau\varepsilon\sum_{n=1}^{N-1}(\||\xi^{n+1}|\|^2+\||\xi^{n-1}|\|^2)\end{aligned} \tag{24}$$

将式(24)代入式(21),可得

$$\begin{aligned}&\|d_t\eta^N\|_M^2+2\|d_t\eta^{N-1}\|_M^2+c_0\||\xi^N|\|^2+c_0\||\xi^{N-1}|\|^2\leqslant\\&C(\tau^4+h^4+k^4)+\varepsilon(\||\xi^N|\|^2+\||\xi^{N-1}|\|^2)+\\&\tau\varepsilon\sum_{n=1}^{N-1}(\||\xi^{n+1}|\|^2+\||\xi^{n-1}|\|^2)\end{aligned} \tag{25}$$

取适当小的 ε,再根据离散的 Gronwall 不等式可得

$$\|d_t\eta^N\|_M^2+2\|d_t\eta^{N-1}\|_M^2+\||\xi^N|\|^2+\||\xi^{N-1}|\|^2\leqslant C(\tau^4+h^4+k^4) \tag{26}$$

根据引理 3 可得

$$\|\eta^N\|_M^2\leqslant\|d_x\eta^N\|_x^2+\|d_y\eta^N\|_y^2\leqslant\||\xi^N|\|^2$$

综上可得出如下定理。

定理　设 p 和 P^n 分别是式(1)—式(3)和式(6)—式(8)的解,$\boldsymbol{u}$ 和 $\boldsymbol{U}^n$ 分别是式(4)—式(5)和式(6)—式(8)的解,网格是正规剖分的,则存在与时间步长 τ 空间步长 h,k 无关的常数 C,对任意 $0\leqslant n<N(N=[T/\tau])$,有

$$\| P^n-p(t^n)\| \leqslant \|\eta^n\| + \|\alpha^n\| \leqslant C(h^2+k^2+\tau^2)$$

$$\| \boldsymbol{U}^n-\boldsymbol{u}(t^n)\| \leqslant \|\xi^n\| + \|\beta^n\| \leqslant C(h^2+k^2+\tau^2)$$

3　数值结果

考虑初边值问题:

$$\frac{\partial^2 p}{\partial t^2}-\frac{\partial^2 p}{\partial x^2}=f(x,t)\qquad (x,t)\in(0,1)\times(0,T]$$

$$p(0,t)=0\qquad p(1,t)=0\qquad t\in(0,T]$$

$$p(x,0)=p^0(x)\qquad \frac{\partial p}{\partial t}(x,0)=q^0(x)\qquad x\in(0,1)$$

其中,$p^0(x)=1-\cos 2\pi x$,$q^0(x)=-(1-\cos 2\pi x)$,$u=-2\pi e^{-t}\sin 2\pi x$。

设上问题的解为 $p(x,t)=e^{-t}(1-\cos 2\pi x)$,而 $f(x,t)$ 是相应的右端。计算格式为

$$\delta_t^2 Z_i^n+[D_xW^x]_i^n=f_i^n\qquad (i=1,2,\cdots,N_x)$$

$$W_{i+\frac{1}{2}}^{n+1}=-[d_xZ]_{i+\frac{1}{2}}^{n+1}\qquad (i=1,2,\cdots,N_x)$$

$$W_{\frac{1}{2}}^{n}=0\qquad W_{N_x+\frac{1}{2}}^{n}=0\qquad n\tau\leqslant T$$

$$Z_i^0=p_i^0\qquad Z_i^1=p_i^0+\tau q_i^0\qquad (i=1,2,\cdots,N_x)$$

令 $x_0=x_{\frac{1}{2}}$,$x_{N_x+1}=x_{N_x+\frac{1}{2}}$,$h_i=x_{i+\frac{1}{2}}-x_{i-\frac{1}{2}}$,$h_{i-\frac{1}{2}}=\frac{h_i+h_{i-1}}{2}$,$[D_xW]_i=\frac{W_{i+\frac{1}{2}}-W_{i-\frac{1}{2}}}{h_i}$,$[d_xZ]_{i+\frac{1}{2}}=\frac{Z_{i+1}-Z_i}{h_{i+\frac{1}{2}}}$,定义

$$P_E=\left\{\sum_{i=1}^{N_x-1}h_i[Z_i^n-p(x_i,n\tau)]^2\right\}^{\frac{1}{2}}$$

$$V_E=\left\{\sum_{i=0}^{N_x-1}h_{i+\frac{1}{2}}\left[W_{i+\frac{1}{2}}^n-u\left(x_{i+\frac{1}{2}},n\tau\right)\right]^2\right\}^{\frac{1}{2}}$$

$$W_1=\max_i\left|\frac{Z_i^n-p(x_i,n\tau)}{p(x_i,n\tau)}\right|$$

$$W_2=\max_i\left|\frac{W_{i+\frac{1}{2}}^n-u(x_{i+\frac{1}{2}},n\tau)}{u(x_{i+\frac{1}{2}},n\tau)}\right|\qquad (i=1,2,\cdots,N_x)$$

取 $N_x=100$,$h_i=h=\frac{1}{N_x}$,$\tau=h$,计算结果如表 1 所示。

表 1　误差比较

Table 1　Comparison of the error

t/s	1.0	2.0	3.0	4.0
P_E	3.70×10^{-4}	2.37×10^{-4}	2.74×10^{-4}	2.56×10^{-4}
V_E	8.05×10^{-3}	8.85×10^{-3}	7.31×10^{-3}	7.46×10^{-3}
W_1	3.88×10^{-6}	9.06×10^{-5}	1.02×10^{-4}	6.74×10^{-5}
W_2	1.36×10^{-5}	1.31×10^{-4}	1.05×10^{-4}	4.54×10^{-4}

4　结论

本文将块中心差分方法和抛物型的 Crank-Nicolson 格式结合起来,在非等距剖分的网格上得了近似解和解的一阶导数。该方法的优越性在于不仅可以求出近似解还可以求出近似解的导数,且解的一阶导数和解具有同样的精度。此方法对分布不均的问题误差也能达到最优。

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(责任编辑　代丽)