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应力函数法计算角度非均质材料切口奇异应力场*

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摘　要　提出了一种将有限元分析与应力函数相结合的方法，来确定角度非均质材料 V 形切口结构的奇异应力场. 首先，基于奇异性特征分析，得到角度非均质材料切口的应力奇性指数. 然后，通过引入 Williams 渐近展开表示的应力函数，将角度非均质材料控制方程和相容方程转化为常微分方程，求解获得应力函数的具体表达式. 进而，由有限元应力结果计算出应力函数渐近展开中的系数. 最后，重构出角度非均质材料切口尖端附近奇异应力场. 论文分析了选取的有限元节点数，特征距离和截断项数对应力强度因子计算结果的影响. 选取不同的有限元节点数时，应力强度因子呈水平线，说明有限元节点数的选取不影响计算结果的稳定性. 当截断项数较少时，应力强度因子随着特征距离的增大而逐渐变化，而当截断项数取到五至六项时，应力强度因子随着特征距离的增大而保持稳定.

关键词　应力函数，角度非均质材料，切口，渐近展开，应力强度因子

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0　引言

角度非均质材料在工程结构中得到了广泛的应用. 对于各向异性材料、压电材料和磁电材料等复合材料，在主轴坐标系下测量它们的材料常数，经过坐标变换后是角度坐标的函数，其力学性能随角度坐标的变化而变化，因此可以看作是一类角度非均质弹性材料. 在角度非均质材料制造和服役过程中，不可避免地会出现裂纹、切口等缺陷. V 形切口尖端附近存在应力奇点，这将引发裂纹并使结构破坏[1,2].

应力函数法[3]能够直接求解应力分量，具有精度高的特点，是弹性理论中的常用解法. Carpinteri 和 Paggi[4]利用 Airy 应力函数并通过特征函数展开法推导出了角度非均质材料切口的渐近应力场公式. 戴耀等[5]从理论上探讨了将均匀材料的特征函数解即 Williams 解推广到非均匀材料的合理性，给出了反平面问题任意阶特征函数的定解方程，并提供了非均匀材料反平面问题的 Williams 型解，从理论上揭示了非均匀性对裂纹尖端场渐近展开式的影响. 燕秀发和戴耀[6]从应力函数出发，推导出了功能梯度材料中裂纹尖端高阶渐近应力场的解析式，并分析了非均匀性参数对裂纹尖端渐近应力场的影响. Theotokoglou 等[7]推导了角度非均匀材料 V 形切口中应力场和位移场的解析解. 目前已有研究仅给出了非均质材料切口应力函数的初步解，切口尖端复杂应力场的解答需进一步探究.

本文提出一种将有限元分析与应力函数法相结合的方法，来确定角度非均质材料 V 形切口结构的奇异应力场. 首先，通过奇异性特征分析得到角度非均质材料切口的应力奇性指数. 然后，通过引入 Williams 渐近展开表示的应力函数，将角度非均质材料的控制方程和相容方程转化为常微分方程. 之后代入角度非均质材料切口的应力奇性指数，通过求解所建立的常微分方程得到应力函数的解析表达式. 进而由有限元应力结果确定应力函数渐近展开

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式中的系数.最后,重构角度非均质材料切口尖端附近奇异应力场,并得到切口应力强度因子.

1 角度非均质材料的应力函数

1.1 控制方程

讨论如图 1(a)所示的角度非均质材料切口.图 1(b)为切口尖端放大区域,图 1(c)为弹性模量的变化函数,其中角度坐标 θ 变化范围是 $\theta\in[-\pi,\pi]$.假定弹性模量按 θ 坐标呈指数变化,即

$$E(\theta)=E_0\mathrm{e}^{\omega\theta} \tag{1}$$

其中,E_0 是切口角平分线 $\theta=0$ 处的弹性模量,ω 是单位为$[\mathrm{rad}]^{-1}$的非均质材料参数.

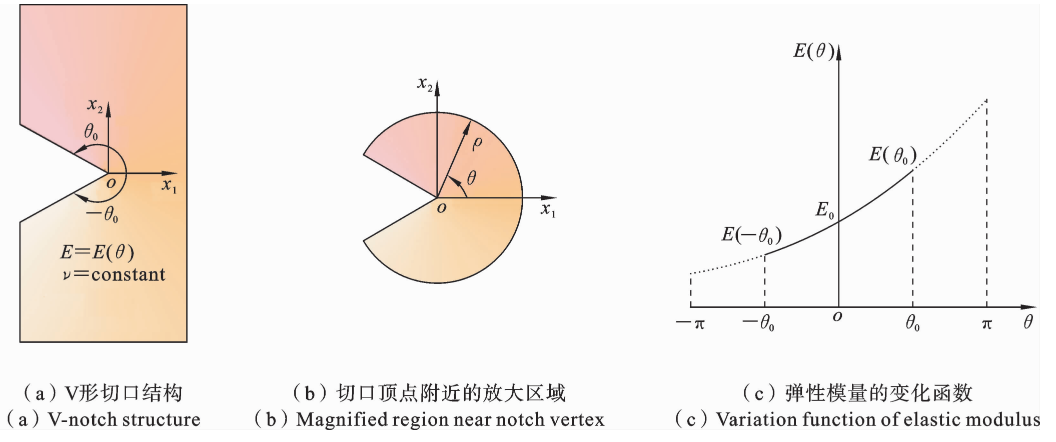


图 1 角度非均质材料切口

Fig. 1 Angularly heterogeneous material notch

考虑平面弹性问题,平面内应力分量可以用极坐标系下的 Ariy 应力函数 F 表示为

$$\begin{cases}\sigma_\rho=\dfrac{1}{\rho}\dfrac{\partial F}{\partial\rho}+\dfrac{1}{\rho^2}\dfrac{\partial^2F}{\partial\theta^2}\\ \sigma_\theta=\dfrac{\partial^2F}{\partial\rho^2}\\ \tau_{\rho\theta}=-\dfrac{1}{\rho}\dfrac{\partial^2F}{\partial\rho\partial\theta}+\dfrac{1}{\rho^2}\dfrac{\partial F}{\partial\theta}\end{cases} \tag{2}$$

对于平面应力状态,有 Hooke 定律

$$\begin{cases}\varepsilon_\rho=\dfrac{1}{E(\theta)}(\sigma_\rho-\nu\sigma_\theta)\\ \varepsilon_\theta=\dfrac{1}{E(\theta)}(\sigma_\theta-\nu\sigma_\rho)\\ \gamma_{\rho\theta}=\dfrac{2(1+\nu)}{E(\theta)}\tau_{\rho\theta}\end{cases} \tag{3}$$

将式(2)代入式(3),再代入如下相容方程

$$\frac{1}{\rho^2}\frac{\partial^2\varepsilon_\rho}{\partial\theta^2}+\frac{\partial^2\varepsilon_\theta}{\partial\rho^2}-\frac{1}{\rho}\frac{\partial\varepsilon_\rho}{\partial\rho}+\frac{2}{\rho}\frac{\partial\varepsilon_\theta}{\partial\rho}=\frac{2}{\rho}\frac{\partial^2\gamma_{\rho\theta}}{\partial\rho\partial\theta}+\frac{2}{\rho^2}\frac{\partial\gamma_{\rho\theta}}{\partial\theta} \tag{4}$$

整理可得,

$$\nabla^4F+\omega^2\left(\frac{1}{\rho^3}\frac{\partial F}{\partial\rho}+\frac{1}{\rho^4}\frac{\partial^2F}{\partial\theta^2}-\frac{\nu}{\rho^2}\frac{\partial^2F}{\partial\rho^2}\right)+\omega\left(\frac{2\nu}{\rho^2}\frac{\partial^3F}{\partial\rho^2\partial\theta}-\frac{2}{\rho^4}\frac{\partial^3F}{\partial\theta^3}-\frac{2}{\rho^3}\frac{\partial^2F}{\partial\rho\partial\theta}\right)+\omega(1+\nu)\left(-\frac{2}{\rho^2}\frac{\partial^3F}{\partial\rho^2\partial\theta}+\frac{2}{\rho^3}\frac{\partial^2F}{\partial\rho\partial\theta}-\frac{2}{\rho^4}\frac{\partial F}{\partial\theta}\right)=0 \tag{5}$$

从式(5)中可以看出,第一项是双谐波项,由于非均匀性参数 ω,出现了附加的低微分项,当 $\omega=0$ 时,式(5)对应于均质材料应力函数的重调和条件.类似的推导过程可参见文献[4,5].式(5)的解将通过下一小节渐近分析方法得到.

1.2 应力函数的渐近展开

对于角度非均质材料,切口尖端附近的应力奇异性更为复杂.在求得角度非均质材料切口的应力奇异性指数的基础上,应力函数可以在极坐标下展开[8,9]为

$$F=\sum_{k=1}^{\infty}F_k(\rho,\theta)=\sum_{k=1}^{\infty}\rho^{\lambda_k+2}f_k(\theta) \tag{6}$$

其中,$F_k(\rho,\theta)=\rho^{\lambda_k+2}f_k(\theta)$,$\lambda_k$ 为非均质材料切口的应力奇异性指数[10,11].

将渐近展开式(6)代入式(5),对各阶 λ_k 进行阶次分析,可得

$$f_k^{(4)}(\theta)-2\omega f_k^{(3)}(\theta)+[2(\lambda_k^2+2\lambda_k+2)+\omega^2]f_k^{(2)}(\theta)-2\omega[(\lambda_k+1)^2+(\lambda_k+1)(1-\nu)+1]f'_k(\theta)+[(\lambda_k+2)^2\lambda_k^2-\omega^2(\lambda_k+2)(\lambda_k\nu+\nu-1)]f_k(\theta)=0 \tag{7}$$

其中$(\cdots)'=\mathrm{d}(\cdots)/\mathrm{d}\theta$.

根据式(2),应力分量的各阶分量可以用特征角函数 f 表示为

$$\begin{cases}\sigma_{\rho k}=\rho^{\lambda_k}[f_k^{(2)}+(\lambda_k+2)f_k]\\ \sigma_{\theta k}=\rho^{\lambda_k}(\lambda_k+1)(\lambda_k+2)f_k\\ \tau_{\rho\theta k}=-\rho^{\lambda_k}(\lambda_k+1)f'_k\end{cases} \tag{8}$$

1.3　应力特征函数的求解

下面来研究微分方程(7)的求解,与之对应的特征方程为

$$r^4-2\omega r^3+[2(\lambda_k^2+2\lambda_k+2)+\omega^2]r^2-2\omega[(\lambda_k+1)^2+(\lambda_k+1)(1-\nu)+1]r+[(\lambda_k+2)^2\lambda_k^2-\omega^2(\lambda_k+2)(\lambda_k\nu+\nu-1)]=0 \tag{9}$$

该方程可以用 Mathematica 代码求解,可以考虑下列情况:

(1) 两对共轭复根:$r_i=r_{i\mathrm{R}k}\pm \mathrm{i}r_{i\mathrm{I}k}$,$(i=1,2)$,其中 $\mathrm{i}=\sqrt{-1}$,下标 R 和 I 分别代表复数的实部和虚部,此时 $f_k(\theta)$的通解为

$$f_k(\theta)=[\mathrm{e}^{r_{1\mathrm{R}k}\theta}(A_k\cos r_{1\mathrm{I}k}\theta+B_k\sin r_{1\mathrm{I}k}\theta)+\mathrm{e}^{r_{1\mathrm{R}k}\theta}(C_k\cos r_{2\mathrm{I}k}\theta+D_k\sin r_{2\mathrm{I}k}\theta)] \tag{10}$$

则可以得到极坐标系下的应力分量

$$\sigma_{\rho k}=(\lambda_k+2)\rho^{\lambda_k}[\mathrm{e}^{r_{1\mathrm{R}k}\theta}(A_k\cos r_{1\mathrm{I}k}\theta+B_k\sin r_{1\mathrm{I}k}\theta)+\mathrm{e}^{r_{2\mathrm{R}k}\theta}(C_k\cos r_{2\mathrm{I}k}\theta+D_k\sin r_{2\mathrm{I}k}\theta)]-\rho^{\lambda_k}[\mathrm{e}^{r_{1\mathrm{R}k}\theta}(A_kr_{1\mathrm{I}k}^2\cos r_{1\mathrm{I}k}\theta+B_kr_{1\mathrm{I}k}^2\sin r_{1\mathrm{I}k}\theta)+\mathrm{e}^{r_{2\mathrm{R}k}\theta}(C_kr_{2\mathrm{I}k}^2\cos r_{2\mathrm{I}k}\theta+D_kr_{2\mathrm{I}k}^2\sin r_{2\mathrm{I}k}\theta)]+\rho^{\lambda_k}[2r_{1\mathrm{R}k}\mathrm{e}^{r_{1\mathrm{R}k}\theta}(-A_kr_{1\mathrm{I}k}\sin r_{1\mathrm{I}k}\theta+B_kr_{1\mathrm{I}k}\cos r_{1\mathrm{I}k}\theta)+2r_{2\mathrm{R}k}\mathrm{e}^{r_{2\mathrm{R}k}\theta}(-C_kr_{2\mathrm{I}k}\sin r_{2\mathrm{I}k}\theta+D_kr_{2\mathrm{I}k}\cos r_{2\mathrm{I}k}\theta)]+\rho^{\lambda_k}[r_{1\mathrm{R}k}^2\mathrm{e}^{r_{1\mathrm{R}k}\theta}(A_k\cos r_{1\mathrm{I}k}\theta+B_k\sin r_{1\mathrm{I}k}\theta)+r_{2\mathrm{R}k}^2\mathrm{e}^{r_{2\mathrm{R}k}\theta}(C_k\cos r_{2\mathrm{I}k}\theta+D_k\sin r_{2\mathrm{I}k}\theta)] \tag{11a}$$

$$\sigma_{\theta k}=(\lambda_k+2)(\lambda_k+1)\rho^{\lambda_k}[\mathrm{e}^{r_{1\mathrm{R}k}\theta}(A_k\cos r_{1\mathrm{I}k}\theta+B_k\sin r_{1\mathrm{I}k}\theta)+\mathrm{e}^{r_{2\mathrm{R}k}\theta}(C_k\cos r_{2\mathrm{I}k}\theta+D_k\sin r_{2\mathrm{I}k}\theta)] \tag{11b}$$

$$\tau_{\rho\theta k}=-(\lambda_k+1)\rho^{\lambda_k}[\mathrm{e}^{r_{1\mathrm{R}k}\theta}(-A_kr_{1\mathrm{I}n}\sin r_{1\mathrm{I}k}\theta+B_kr_{1\mathrm{I}k}\cos r_{1\mathrm{I}k}\theta)+\mathrm{e}^{r_{2\mathrm{R}k}\theta}(-C_kr_{2\mathrm{I}k}\sin r_{2\mathrm{I}k}\theta+D_kr_{2\mathrm{I}k}\cos r_{2\mathrm{I}k}\theta)]-(\lambda_k+1)\rho^{\lambda_k}[r_{1\mathrm{R}k}\mathrm{e}^{r_{1\mathrm{R}k}\theta}(A_k\cos r_{1\mathrm{I}k}\theta+B_k\sin r_{1\mathrm{I}k}\theta)+r_{2\mathrm{R}k}\mathrm{e}^{r_{2\mathrm{R}k}\theta}(C_k\cos r_{2\mathrm{I}k}\theta+D_k\sin r_{2\mathrm{I}k}\theta)] \tag{11c}$$

化简得到

$$\sigma_{\rho k}=A_k\rho^{\lambda_k}\mathrm{e}^{r_{1\mathrm{R}k}\theta}[(\lambda_k+2-r_{1\mathrm{I}k}^2+r_{1\mathrm{R}k}^2)\cos r_{1\mathrm{I}k}\theta-2r_{1\mathrm{R}k}r_{1\mathrm{I}k}\sin r_{1\mathrm{I}k}\theta]+B_k\rho^{\lambda_k}\mathrm{e}^{r_{1\mathrm{R}k}\theta}[(\lambda_k+2-r_{1\mathrm{I}k}^2+r_{1\mathrm{R}k}^2)\sin r_{1\mathrm{I}k}\theta+2r_{1\mathrm{R}k}r_{1\mathrm{I}k}\cos r_{1\mathrm{I}k}\theta]+C_k\rho^{\lambda_k}\mathrm{e}^{r_{2\mathrm{R}k}\theta}[(\lambda_k+2-r_{2\mathrm{I}k}^2+r_{2\mathrm{R}k}^2)\cos r_{2\mathrm{I}k}\theta-2r_{2\mathrm{R}k}r_{2\mathrm{I}k}\sin r_{2\mathrm{I}k}\theta]+D_k\rho^{\lambda_k}\mathrm{e}^{r_{2\mathrm{R}k}\theta}[(\lambda_k+2-r_{2\mathrm{I}k}^2+r_{2\mathrm{R}k}^2)\sin r_{2\mathrm{I}k}\theta+2r_{2\mathrm{R}k}r_{2\mathrm{I}k}\cos r_{2\mathrm{I}k}\theta] \tag{12a}$$

$$\sigma_{\theta k}=(\lambda_k+2)(\lambda_k+1)\rho^{\lambda_k}[A_n\mathrm{e}^{r_{1\mathrm{R}k}\theta}\cos r_{1\mathrm{I}k}\theta+B_n\mathrm{e}^{r_{1\mathrm{R}k}\theta}\sin r_{1\mathrm{I}k}\theta]+(\lambda_k+2)(\lambda_k+1)\rho^{\lambda_k}[C_n\mathrm{e}^{r_{2\mathrm{R}k}\theta}\cos r_{2\mathrm{I}k}\theta+D_n\mathrm{e}^{r_{2\mathrm{R}k}\theta}\sin r_{2\mathrm{I}k}\theta] \tag{12b}$$

$$\tau_{\rho\theta k}=-(\lambda_k+1)\rho^{\lambda_k}\mathrm{e}^{r_{1\mathrm{R}k}\theta}[A_k(-r_{1\mathrm{I}k}\sin r_{1\mathrm{I}k}\theta+r_{1\mathrm{R}k}\cos r_{1\mathrm{I}k}\theta)+B_k(r_{1\mathrm{I}k}\cos r_{1\mathrm{I}k}\theta+r_{1\mathrm{R}k}\sin r_{1\mathrm{I}k}\theta)]-(\lambda_k+1)\rho^{\lambda_k}\mathrm{e}^{r_{2\mathrm{R}k}\theta}[C_k(-r_{2\mathrm{I}k}\sin r_{2\mathrm{I}k}\theta+r_{2\mathrm{R}k}\cos r_{2\mathrm{I}k}\theta)+D_k(r_{2\mathrm{I}k}\cos r_{2\mathrm{I}k}\theta+r_{2\mathrm{R}k}\sin r_{2\mathrm{I}k}\theta)] \tag{12c}$$

(2) 一对共轭复根和两个不相等的实根:$r_1=r_{1\mathrm{R}k}\pm \mathrm{i}r_{1\mathrm{I}k}$,$r_{2\mathrm{R}k}$和 $r_{3\mathrm{R}k}$,此时 $f_k(\theta)$的通解为

$$f_k(\theta)=\mathrm{e}^{r_{1\mathrm{R}k}\theta}(A_k\cos r_{1\mathrm{I}k}\theta+B_k\sin r_{1\mathrm{I}k}\theta)+C_k\mathrm{e}^{r_{2\mathrm{R}k}\theta}+D_k\mathrm{e}^{r_{3\mathrm{R}k}\theta} \tag{13}$$

对应的应力分量为

$$\sigma_{\rho k}=(\lambda_k+2)\rho^{\lambda_k}[\mathrm{e}^{r_{1\mathrm{R}k}\theta}(A_k\cos r_{1\mathrm{I}k}\theta+B_k\sin r_{1\mathrm{I}k}\theta)+C_k\mathrm{e}^{r_{2\mathrm{R}k}\theta}+D_k\mathrm{e}^{r_{3\mathrm{R}k}\theta}]+\rho^{\lambda_k}[r_{1\mathrm{R}k}^2\mathrm{e}^{r_{1\mathrm{R}k}\theta}(A_k\cos r_{1\mathrm{I}k}\theta+B_k\sin r_{1\mathrm{I}k}\theta)+C_kr_{2\mathrm{R}k}^2\mathrm{e}^{r_{2\mathrm{R}k}\theta}+D_kr_{3\mathrm{R}k}^2\mathrm{e}^{r_{3\mathrm{R}k}\theta}]+\rho^{\lambda_k}[2r_{1\mathrm{R}k}r_{1\mathrm{I}k}\mathrm{e}^{r_{1\mathrm{R}k}\theta}(-A_k\sin r_{1\mathrm{I}k}\theta+B_k\cos r_{1\mathrm{I}k}\theta)]-\rho^{\lambda_k}[r_{1\mathrm{I}k}^2\mathrm{e}^{r_{1\mathrm{R}k}\theta}(A_k\cos r_{1\mathrm{I}k}\theta+B_k\sin r_{1\mathrm{I}k}\theta)] \tag{14a}$$

$$\sigma_{\theta k}=(\lambda_k+2)(\lambda_k+1)\rho^{\lambda_k}[\mathrm{e}^{r_{1\mathrm{R}k}\theta}(A_k\cos r_{1\mathrm{I}k}\theta+B_k\sin r_{1\mathrm{I}k}\theta)+C_k\mathrm{e}^{r_{2\mathrm{R}k}\theta}+D_k\mathrm{e}^{r_{3\mathrm{R}k}\theta}] \tag{14b}$$

$$\tau_{\rho\theta k}=-(\lambda_k+1)\rho^{\lambda_k}[r_{1\mathrm{R}k}\mathrm{e}^{r_{1\mathrm{R}k}\theta}(A_k\cos r_{1\mathrm{I}k}\theta+B_k\sin r_{1\mathrm{I}k}\theta)+C_kr_{2\mathrm{R}k}\mathrm{e}^{r_{2\mathrm{R}k}\theta}+D_kr_{3\mathrm{R}k}\mathrm{e}^{r_{3\mathrm{R}k}\theta}]-(\lambda_k+1)\rho^{\lambda_k}[\mathrm{e}^{r_{1\mathrm{R}k}\theta}(-A_kr_{1\mathrm{I}k}\sin r_{1\mathrm{I}k}\theta+B_kr_{1\mathrm{I}k}\cos r_{1\mathrm{I}k}\theta)] \tag{14c}$$

化简得到

$$\sigma_{\rho k}=A_k\rho^{\lambda_k}\mathrm{e}^{r_{1\mathrm{R}k}\theta}[(\lambda_k+2-r_{1\mathrm{I}k}^2+r_{1\mathrm{R}k}^2)\cos r_{1\mathrm{I}k}\theta-2r_{1\mathrm{R}k}r_{1\mathrm{I}k}\sin r_{1\mathrm{I}k}\theta]+B_k\rho^{\lambda_k}\mathrm{e}^{r_{1\mathrm{R}k}\theta}[(\lambda_k+2-r_{1\mathrm{I}k}^2+r_{1\mathrm{R}k}^2)\sin r_{1\mathrm{I}k}\theta+2r_{1\mathrm{R}k}r_{1\mathrm{I}k}\cos r_{1\mathrm{I}k}\theta]+C_k\rho^{\lambda_k}\mathrm{e}^{r_{2\mathrm{R}k}\theta}(\lambda_k+2+r_{2\mathrm{R}k}^2)+D_k\rho^{\lambda_k}\mathrm{e}^{r_{3\mathrm{R}k}\theta}(\lambda_k+2+r_{3\mathrm{R}k}^2) \tag{15a}$$

$$\sigma_{\theta k}=(\lambda_k+2)(\lambda_k+1)\rho^{\lambda_k}[A_n\mathrm{e}^{r_{1\mathrm{R}k}\theta}\cos r_{1\mathrm{I}k}\theta+B_n\mathrm{e}^{r_{1\mathrm{R}k}\theta}\sin r_{1\mathrm{I}k}\theta]+(\lambda_k+2)(\lambda_k+1)\rho^{\lambda_k}(C_n\mathrm{e}^{r_{2\mathrm{R}k}\theta}+D_n\mathrm{e}^{r_{3\mathrm{R}k}\theta}) \tag{15b}$$

$$\tau_{\rho\theta k}=-(\lambda_k+1)\rho^{\lambda_k}\mathrm{e}^{r_{1\mathrm{R}k}\theta}[A_k(-r_{1\mathrm{I}k}\sin r_{1\mathrm{I}k}\theta+r_{1\mathrm{R}k}\cos r_{1\mathrm{I}k}\theta)+B_k(r_{1\mathrm{I}k}\cos r_{1\mathrm{I}k}\theta+r_{1\mathrm{R}k}\sin r_{1\mathrm{I}k}\theta)]-(\lambda_k$$

$$+1)\rho^{\lambda_k}(C_k e^{r_{2Rk}\theta}r_{2Rk}+D_k e^{r_{3Rk}\theta}r_{3Rk}) \tag{15c}$$

(3) 一对共轭复根和两个相等的实根：$r_1=r_{1Rk}\pm i r_{1Ik}$ 和 $r_{3Rk}=r_{2Rk}$，此时 $f_k(\theta)$ 的通解为

$$f_k(\theta)=e^{r_{1Rk}\theta}(A_k\cos r_{1Ik}\theta+B_k\sin r_{1Ik}\theta)+C_k e^{r_{2Rk}\theta}+D_k\theta e^{r_{2Rk}\theta} \tag{16}$$

对应的应力分量为

$$\sigma_{\rho k}=(\lambda_k+2)\rho^{\lambda_k}[e^{r_{1Rk}\theta}(A_k\cos r_{1Ik}\theta+B_k\sin r_{1Ik}\theta)+C_k e^{r_{2Rk}\theta}+D_k\theta e^{r_{2Rk}\theta}]+\rho^{\lambda_k}[r_{1Rk}^2 e^{r_{1Rk}\theta}(A_k\cos r_{1Ik}\theta+B_k\sin r_{1Ik}\theta)+C_k r_{2Rk}^2 e^{r_{2Rk}\theta}+D_k r_{2Rk}^2\theta e^{r_{2Rk}\theta}]+\rho^{\lambda_k}[2r_{1Rk}r_{1Ik}e^{r_{1Rk}\theta}(-A_k\sin r_{1Ik}\theta+B_k\cos r_{1Ik}\theta)+2D_k r_{2Rk}e^{r_{2Rk}\theta}]-\rho^{\lambda_k}[r_{1Ik}^2 e^{r_{1Rk}\theta}(A_k\cos r_{1Ik}\theta+B_k\sin r_{1Ik}\theta)] \tag{17a}$$

$$\sigma_{\theta k}=(\lambda_k+2)(\lambda_k+1)\rho^{\lambda_k}[e^{r_{1Rk}\theta}(A_k\cos r_{1Ik}\theta+B_k\sin r_{1Ik}\theta)+C_k e^{r_{2Rk}\theta}+D_k\theta e^{r_{2Rk}\theta}] \tag{17b}$$

$$\tau_{\rho\theta k}=-(\lambda_k+1)\rho^{\lambda_k}[r_{1Rk}e^{r_{1Rk}\theta}(A_k\cos r_{1Ik}\theta+B_k\sin r_{1Ik}\theta)+C_k r_{2Rk}e^{r_{2Rk}\theta}+D_k r_{3Rk}\theta e^{r_{3Rk}\theta}+D_k e^{r_{3Rk}\theta}]-(\lambda_k+1)\rho^{\lambda_k}[e^{r_{1Rk}\theta}(-A_k r_{1Ik}\sin r_{1Ik}\theta+B_k r_{1Ik}\cos r_{1Ik}\theta)] \tag{17c}$$

化简得到

$$\sigma_{\rho k}=A_k\rho^{\lambda_k}e^{r_{1Rk}\theta}[(\lambda_k+2-r_{1Ik}^2+r_{1Rk}^2)\cos r_{1Ik}\theta-2r_{1Rk}r_{1Ik}\sin r_{1Ik}\theta]+B_k\rho^{\lambda_k}e^{r_{1Rk}\theta}[(\lambda_k+2-r_{1Ik}^2+r_{1Rk}^2)\sin r_{1Ik}\theta+2r_{1Rk}r_{1Ik}\cos r_{1Ik}\theta]+C_k\rho^{\lambda_k}e^{r_{2Rk}\theta}(\lambda_k+2+r_{2Rk}^2)+D_k\rho^{\lambda_k}e^{r_{2Rk}\theta}[(\lambda_k+2)\theta+r_{2Rk}^2\theta+2r_{2Rk}] \tag{18a}$$

$$\sigma_{\theta k}=(\lambda_k+2)(\lambda_k+1)\rho^{\lambda_k}[A_n e^{r_{1Rk}\theta}\cos r_{1Ik}\theta+B_n e^{r_{1Rk}\theta}\sin r_{1Ik}\theta]+(\lambda_k+2)(\lambda_k+1)\rho^{\lambda_k}(C_n e^{r_{2Rk}\theta}+D_n\theta e^{r_{2Rk}\theta}) \tag{18b}$$

$$\tau_{\rho\theta k}=-(\lambda_k+1)\rho^{\lambda_k}e^{r_{1Rk}\theta}[A_k(-r_{1Ik}\sin r_{1Ik}\theta+r_{1Rk}\cos r_{1Ik}\theta)+B_k(r_{1Ik}\cos r_{1Ik}\theta+r_{1Rk}\sin r_{1Ik}\theta)]-(\lambda_k+1)\rho^{\lambda_k}[C_k e^{r_{2Rk}\theta}r_{2Rk}+D_k e^{r_{2Rk}\theta}(r_{2Rk}\theta+1)] \tag{18c}$$

上述所列情况，将非均质材料切口尖端的应力分量表达出来，其中只有每一阶的幅值系数为未知量，下面研究幅值系数的求解工作.

2　应力函数幅值系数的计算

下面基于有限元结果计算角度非均质材料切口应力函数的幅值系数. 然后，通过选择不同阶的幅值系数，分别对奇异应力和高阶非奇异应力进行重构.

计算 V 形切口结构应力场的有限元网格如图 2(a)所示，其中网格类型选择八节点四边形单元. 为确保有限元结算结果的稳定性和准确性，在切口尖端划分如图 2(b)所示的放射性网格，每个圆环被均等划分为 80 份，远离切口的位置划分稀疏的均匀网格. 选择切口尖端附近 n 个节点上的应力(若 $n>81$，则额外选择相邻圆环上的节点)，如图 2(b)所示，用于求解渐近展开式中的幅值系数. 所选取节点的径向坐标 ρ 称为渐近展开式(6)中的特征距离，即节点距离原点的长度. 边界条件的设置和后续具体问题一致.

经坐标变换，有限元分析得到的切口顶点附近节点的应力结果($\sigma_x,\sigma_y,\tau_{xy}$)可转换到极坐标系下

$$\begin{cases}\sigma_\rho=\dfrac{\sigma_x+\sigma_y}{2}+\dfrac{\sigma_x-\sigma_y}{2}\cos 2\theta+\tau_{xy}\sin 2\theta\\ \sigma_\theta=\dfrac{\sigma_x+\sigma_y}{2}-\dfrac{\sigma_x-\sigma_y}{2}\cos 2\theta-\tau_{xy}\sin 2\theta\\ \tau_{\rho\theta}=\dfrac{\sigma_y-\sigma_x}{2}\sin 2\theta+\tau_{xy}\cos 2\theta\end{cases} \tag{19}$$

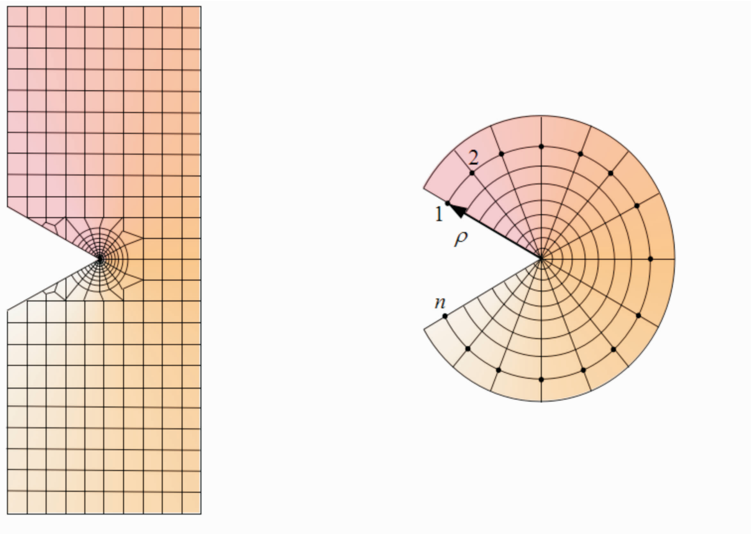
将应力结果式(19)代入式(8)，可得

$$\begin{Bmatrix}\sigma_\rho\\ \sigma_\theta\\ \tau_{\rho\theta}\end{Bmatrix}=\begin{Bmatrix}\sum_{k=1}^{N}\sigma_{\rho k}(A_k,B_k,C_k,D_k)\\ \sum_{k=1}^{N}\sigma_{\theta k}(A_k,B_k,C_k,D_k)\\ \sum_{k=1}^{N}\tau_{\rho\theta k}(A_k,B_k,C_k,D_k)\end{Bmatrix} \tag{20}$$

若选取图 2(b)中 n 个节点的应力，则式(20)可展开为

$$\begin{Bmatrix}\sigma_{\rho 1}\\ \sigma_{\theta 1}\\ \tau_{\rho\theta 1}\\ \vdots\\ \sigma_{\rho n}\\ \sigma_{\theta n}\\ \tau_{\rho\theta n}\end{Bmatrix}=\begin{bmatrix}f_{11}^A & f_{11}^B & f_{11}^C & f_{11}^D & \cdots & f_{1N}^A & f_{1N}^B & f_{1N}^C & f_{1N}^D\\ g_{11}^A & g_{11}^B & g_{11}^C & g_{11}^D & \cdots & g_{1N}^A & g_{1N}^B & g_{1N}^C & g_{1N}^D\\ h_{11}^A & h_{11}^B & h_{11}^C & h_{11}^D & \cdots & h_{1N}^A & h_{1N}^B & h_{1N}^C & h_{1N}^D\\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots\\ f_{n1}^A & f_{n1}^B & f_{n1}^C & f_{n1}^D & \cdots & f_{nN}^A & f_{nN}^B & f_{nN}^C & f_{nN}^D\\ g_{n1}^A & g_{n1}^B & g_{n1}^C & g_{n1}^D & \cdots & g_{nN}^A & g_{nN}^B & g_{nN}^C & g_{nN}^D\\ h_{n1}^A & h_{n1}^B & h_{n1}^C & h_{n1}^D & \cdots & h_{nN}^A & h_{nN}^B & h_{nN}^C & h_{nN}^D\end{bmatrix}\begin{Bmatrix}A_1\\ B_1\\ C_1\\ D_1\\ \vdots\\ A_N\\ B_N\\ C_N\\ D_N\end{Bmatrix} \tag{21}$$

其中，$f_{jk}^*,g_{jk}^*,h_{jk}^*$ ($j=1,\cdots,n,\ k=1,\cdots,N,\ *=A,B,C,D$)是式(20)中的系数.



（a）整体网格
（a）Total mesh

（b）计算幅值系数的节点
（b）Nodes for determining amplitude coefficients

图 2　有限元网格

Fig. 2　Finite element mesh

式(21)可以简写为

$$\{S\}_{3n\times1}=[C]_{3n\times4N}\{A\}_{4N\times1}\tag{22}$$

现在将图 2(b)所示节点上的有限元应力分量代入式(22)求解未知系数. 已知应力分量的个数 $3n$ 通常大于未知渐近展开式幅值系数的个数 $4N$. 因此,方程式(22)是一组超定方程,幅值系数可以用如下最小二乘方法求解

$$\{A\}=([C]^{\mathrm T}[C])^{-1}[C]^{\mathrm T}\{S\}\tag{23}$$

将式(23)得到的幅值系数和奇异性指数重新代入式(8),得到切口尖端附近的渐近应力场. 根据式(8)中截断的序列项,可以依次求出奇异应力项和非奇异应力项.

得到系数 A_k,B_k,C_k 和 D_k 之后,根据前两阶奇异性指数和应力函数,可以计算出 V 形切口的广义应力强度因子. 如果前两阶奇异性指数为实数,则切口应力强度因子可定义为

$$K_{\mathrm I}=\lim_{\rho\to0}\sqrt{2\pi}\rho^{\lambda_1}\sigma_{\theta1}|_{\theta=0}=\sqrt{2\pi}(\lambda_1+2)(\lambda_1+1)[A_1\mathrm e^{r_{1\mathrm R1}\theta}\cos r_{1\mathrm I1}\theta+B_1\mathrm e^{r_{1\mathrm R1}\theta}\sin r_{1\mathrm I1}\theta]_{\theta=0}+\sqrt{2\pi}(\lambda_1+2)(\lambda_1+1)[C_1\mathrm e^{r_{2\mathrm R1}\theta}\cos r_{2\mathrm I1}\theta+D_1\mathrm e^{r_{2\mathrm R1}\theta}\sin r_{2\mathrm I1}\theta]_{\theta=0}=\sqrt{2\pi}(\lambda_1+2)(\lambda_1+1)(A_1+C_1)\tag{24a}$$

$$K_{\mathrm{II}}=\lim_{\rho\to0}\sqrt{2\pi}\rho^{\lambda_1}\tau_{\rho\theta2}|_{\theta=0}=-\sqrt{2\pi}(\lambda_2+1)\mathrm e^{r_{1\mathrm R2}\theta}[A_2(-r_{1\mathrm I2}\sin r_{1\mathrm I2}\theta+r_{1\mathrm R2}\cos r_{1\mathrm I2}\theta)+B_2(r_{1\mathrm I2}\cos r_{1\mathrm I2}\theta+r_{1\mathrm R2}\sin r_{1\mathrm I2}\theta)]_{\theta=0}-\sqrt{2\pi}(\lambda_2+1)\mathrm e^{r_{2\mathrm R2}\theta}[C_2(-r_{2\mathrm I2}\sin r_{2\mathrm I2}\theta+r_{2\mathrm R2}\cos r_{2\mathrm I2}\theta)+D_2(r_{2\mathrm I2}\cos r_{2\mathrm I2}\theta+r_{2\mathrm R2}\sin r_{2\mathrm I2}\theta)]_{\theta=0}=-\sqrt{2\pi}(\lambda_2+1)(A_2r_{1\mathrm R2}+B_2r_{1\mathrm I2}+C_2r_{2\mathrm R2}+D_2r_{2\mathrm I2})\tag{24b}$$

3　数值算例

3.1　均质材料切口

当式(1)中 $\omega=0$,材料的角度非均质特性退化为均质特性,与微分方程对应的特征方程变为

$$r^4+2(\lambda_k^2+2\lambda_k+2)r^2+(\lambda_k+2)^2\lambda_k^2=0\tag{25}$$

方程有两对共轭复根 $r_1=0\pm\mathrm i\lambda_k$,$r_2=0\pm\mathrm i(\lambda_k+2)$. 考虑到 $r_{1\mathrm Rk}=r_{2\mathrm Rk}=0$,则应力分量

$$\sigma_{\rho k}=\frac{1}{\rho}\frac{\partial F_k}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2F_k}{\partial\theta^2}=\rho^{\lambda_k}(\lambda_k+2-\lambda_k^2)[A_k\cos\lambda_k\theta+B_k\sin\lambda_k\theta]-\rho^{\lambda_k}(\lambda_k+2)(\lambda_k+1)[C_k\cos(\lambda_k+2)\theta+D_k\sin(\lambda_k+2)\theta]\tag{26a}$$

$$\sigma_{\theta k}=\frac{\partial^2F_k}{\partial\rho^2}=\rho^{\lambda_k}(\lambda_k+2)(\lambda_k+1)[A_k\cos\lambda_k\theta+B_k\sin\lambda_k\theta]+\rho^{\lambda_k}(\lambda_k+2)(\lambda_k+1)[C_k\cos(\lambda_k+2)\theta+D_k\sin(\lambda_k+2)\theta]\tag{26b}$$

$$\tau_{\rho\theta k}=-\frac{\partial}{\partial\rho}\left(\frac{1}{\rho}\frac{\partial F_k}{\partial\theta}\right)=\rho^{\lambda_k}(\lambda_k+1)[A_kr_{1\mathrm Ik}\sin\lambda_k\theta-B_kr_{1\mathrm Ik}\cos\lambda_k\theta]+\rho^{\lambda_k}(\lambda_k+1)[C_kr_{2\mathrm Ik}\sin(\lambda_k+2)\theta-$$

$D_k r_{21k} \cos(\lambda_k + 2)\theta]$ (26c)

如图 3 所示，受拉伸载荷的斜 V 形切口板，切口板长 $2h=200$ mm，宽 $W=40$ mm，切口深度为 a，$a/W=0.2$，切口角度为 γ，切口角平分线与 x_1 轴方向的夹角为 β. 弹性模量 $E=3.9\times10^9$ Pa，泊松比 $\nu=0.373$，受到载荷 $\sigma=1$ MPa，按平面应力问题计算.

对于均质材料切口，根据式(23)计算应力渐近展开式(26)的幅值系数，其中选择截断项数 $N=5$，进而根据式(24)得到切口应力强度因子. 表 1 所示为不同开口角度和不同切口倾斜角时均质材料 V 形切口板的应力强度因子，从中可以看出，本文方法的精度非常高，与参考解(Chen[12])的误差最大不超过 2.5%.

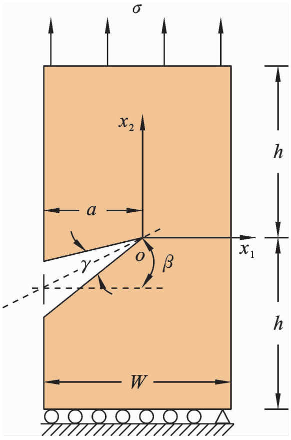


图 3　受拉斜 V 形切口板

Fig. 3　Oblique V-notched plate under tension

表 1　斜切口板的应力强度因子

Table 1　NSIFs for oblique V-notched plate

γ/β(°)	K_{I} ($\mathrm{N\cdot mm^{-2-\lambda_1}}$)			K_{II} ($\mathrm{N\cdot mm^{-2-\lambda_2}}$)		
	Present	Ref.[12]	Δ(%)	Present	Ref.[12]	Δ(%)
30/0	6.9613	6.9017	0.8636	−0.0136	0.0000	/
30/15	6.7367	6.6768	0.8971	−1.2020	−1.2017	0.0250
30/30	6.0870	6.0471	0.6598	−2.2278	−2.1908	1.6889
30/45	5.1137	5.0975	0.3178	−2.8765	−2.8080	2.4395
60/0	6.9735	7.0627	−1.2630	−0.3989E-05	0.0000	/
60/15	6.7367	6.8037	−0.9848	−1.2498	−1.2468	0.2406
60/30	6.0577	6.0705	0.2109	−2.2279	−2.2239	0.1799
60/45	5.0410	4.9854	1.1153	−2.745	−2.7388	0.2264

3.2　非均质材料切口结构

图 4 所示为受拉荷载作用下的对称 V 形切口板，其中切口板的长度 $2h=200$ mm、宽度 $W=40$ mm、深度为 a，荷载 $\sigma=\tau=1.0$ MPa. 切口的开口角度定义为 γ. 非均质材料参数[10]中的弹性模量 $E_0=3.9\times10^9$ Pa、泊松比 $\nu=0.373$、非均质性参数 $\omega=(\ln\kappa)/2\pi$，且 $\kappa=5.0$.

表 2 列出了复合加载模式下，角度非均质材料对称 V 形切口的应力强度因子，从中可以看出，随着切口长度的增加，Ⅰ型应力强度因子 K_{I} 和Ⅱ型应力强度因子 K_{II} 的绝对值逐渐增大. 整体来看，Ⅱ型应力强度因子较小，说明角度非均质材料对于剪切变形

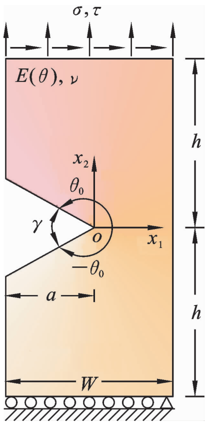


图 4　受拉载荷作用下的对称 V 形切口板

Fig. 4　Symmetric V-notched plate under tension loading

有更高的抵抗力，这可能是由于沿着角度方向均匀变化的材料降低了切口尖端的剪切应力的奇异性.

表 2　复合型荷载下对称 V 形切口的应力强度因子

Table 2　NSIFs of symmetric V-notch under mixed-mode loading

a/W	$\gamma=30°$		$\gamma=60°$		$\gamma=90°$	
	$K_{\mathrm{I}}\,(\mathrm{N\cdot mm^{-2-\lambda_1}})$	$K_{\mathrm{II}}\,(\mathrm{N\cdot mm^{-2-\lambda_2}})$	$K_{\mathrm{I}}\,(\mathrm{N\cdot mm^{-2-\lambda_1}})$	$K_{\mathrm{II}}\,(\mathrm{N\cdot mm^{-2-\lambda_2}})$	$K_{\mathrm{I}}\,(\mathrm{N\cdot mm^{-2-\lambda_1}})$	$K_{\mathrm{II}}\,(\mathrm{N\cdot mm^{-2-\lambda_2}})$
0.1	70.4076	−12.0972	60.8039	−0.9509	63.8598	1.2298
0.2	98.9790	−14.4782	86.9821	0.5967	88.5846	3.1497
0.3	128.9690	−17.0272	114.1478	1.9969	114.4144	4.7558
0.4	168.4420	−21.4540	149.3308	2.8435	148.5497	6.0305
0.5	227.4695	−29.6866	201.3211	2.8264	200.0659	7.0846

取 $a/W=0.3$，$\gamma=30°$，来研究有限元节点数 n、特征距离 ρ 和截断项数 N 对应力强度因子 K 的影响.

图 5 给出了不同的截断项数 N 下，本文方法所计算的应力强度因子 K 随着有限元节点数的变化情况，从中可以看出，随着选取的有限元节点数的不同，应力强度因子 K_{I} 和 K_{II} 的数值计算结果分别呈平线，数值保持稳定，说明有限元节点数的选取不影响计算结果的稳定性.

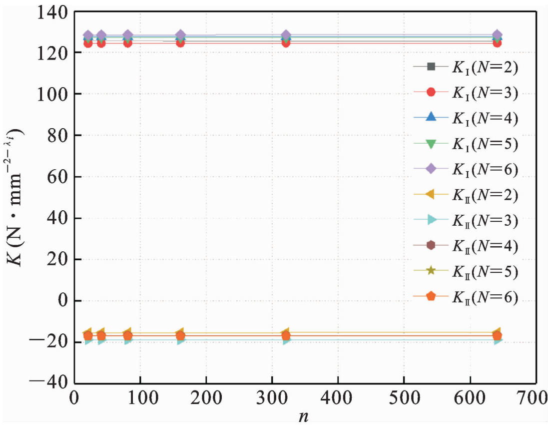


图 5　选取的有限元节点数对应力强度因子的影响

Fig. 5　Effect of selected finite element nodes on NSIFs

图 6 所示为不同的截断项数 N 下，本文方法所计算的应力强度因子 K 随着特征距离的变化情况. 可以看出，当截断项数较少时，应力强度因子 K 的数值计算结果随着特征距离的增大而逐渐变化. 而当截断项数取到五至六项时，应力强度因子 K 的数值计算结果随着特征距离的增大而保持稳定.

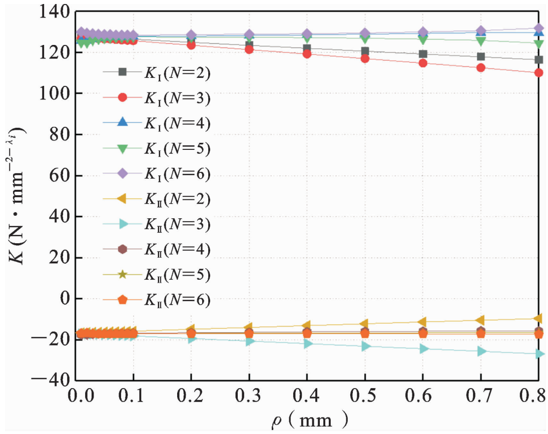


图 6　特征距离对应力强度因子的影响

Fig. 6　Effect of characteristic distance on NSIFs

4　结论

本文给出了非均质材料切口尖端应力场的半解析解，来表达非均质材料切口尖端的奇异应力场. 与已有研究相比，本文给出了非均质材料切口详细的应力表达式，并且可以直接计算切口应力强度因子. 首先，利用奇异性特征分析，得到角度非均质材料切口的应力奇性指数. 然后，通过求解由应力函数渐近展开式导出的常微分方程，得到应力函数的解析表达式. 再结合有限元应力结果确定应力函数展开式的幅值系数. 最后，重构出角度非均质材料切口尖端

附近渐近应力场.

计算了均质材料裂纹、切口的应力强度因子以及角度非均质材料裂纹、切口的应力强度因子. 分析了有限元节点数,特征距离和截断项数对应力强度因子的影响. 随着选取的有限元节点数的不同,应力强度因子呈一条水平线,数值保持稳定,说明有限元节点数的选取不影响计算结果的稳定性. 当截断项数较少时,应力强度因子随着特征距离的增大而逐渐变化,而当截断项数取到五至六项时,应力强度因子随着特征距离的增大而保持稳定.

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Stress Function Method for Singular Stress Calculation near the Vertex of Angularly Heterogeneous Material Notch

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Abstract A novel method combing finite element analysis and a stress function is presented to determine the complete singular stress field in angularly heterogeneous material V-notched structure. This is motivated by the challenge of calculating the stress field, which initiates cracks and structural failure. First, stress singularity orders are obtained through singularity characteristic analysis. Then, the governing and the compatibility equations for the angularly heterogeneous material are transformed into ordinary differential equations using a stress function based on the Williams asymptotic expansion. Solving these yields the stress function expression. Subsequently, coefficients in the asymptotic expansion are determined from finite element stress results, reconstructing the asymptotic stress field near the notch tip. The effects of the number of finite element nodes, characteristic distance, and truncation terms on stress intensity factor calculations are examined. Results show the stress intensity factor stabilizes with an increasing number of finite element nodes, indicating the selection of these nodes does not affect result stability. Stress intensity factors change with characteristic distance when the number of truncation terms is small, but stabilize as the number of truncated terms approaches five or six.

Key words stress function, angularly heterogeneous material, notch, asymptotic expansion, stress intensity factor