

基于近场动力学算子方法的各向异性板自由振动分析*

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摘要 各向异性材料广泛应用于诸多工程领域, 研究其振动特性对结构减振与安全设计具有重要意义. 本文基于一种新的近场动力学算子方法(Peridynamic operator method, PDOM)构建非局部各向异性模型, 并应用于各向异性板的自由振动分析. 该模型结合了 PDOM 将局部微分及其乘积转化为非局部积分的特点, 将经典各向异性理论中的应变能密度从局部形式重构为非局部形式. 同时, 采用变分原理并引入自由振动方程, 构建了适用于各向异性自由振动问题的 PDOM 求解方案. 通过三个数值算例: 各向异性矩形薄板、各向异性矩形含裂纹板、各向异性矩形含孔板的自由振动, 并将计算结果与有限元结果比较, 说明了模型在计算含缺陷以及不连续的各向异性板自由振动问题时具有收敛性、稳定性以及高计算精度.

关键词 自由振动, 各向异性, 近场动力学算子方法, 非局部, 变分原理

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0 引言

各向异性材料广泛应用于航空航天、船舶建造、建筑工程等领域^[1,2]. 了解各向异性材料与结构的振动特性, 进而采取合理有效的减振措施, 具有理论和实践意义. 传统的经典数值方法在处理存在高阶导数和奇异点等情况的各向异性问题时, 通常需要额外处理. 近年来, 研究者尝试采用辛叠加法^[3,4]、伽辽金法^[5]、分离变量法^[6]、微分求积法^[7]和 Rayleigh-Ritz 法^[8,9]等分析各向异性振动问题. 在新兴的数值方法中, 近场动力学(Peridynamics, PD)^[10,11]方法以其在处理非连续力学问题中的优势, 近二十年来被广泛应用于冲击爆炸^[12,13]、流固耦合^[14,15]、水力劈裂^[16,17]和热力耦合^[18,19]等问题模拟. 针对振动问题, Diana^[20]提出了一种正交各向异性微极 PD 模型进行含孔板的自由振动模态分析; 吴林潮等^[21]和 Jafari 等^[22]分别基于 PD 理论分析了欧拉梁和铁木辛柯梁的自由振动特性; Heo 等^[23]应用 PD 方法分析了含裂纹板的自由振动. 在 PD 理

论基础上, 研究者近年又提出一些非局部建模和求解方法, 典型如近场动力学微分算子^[24,25](Peridynamic differential operator, PDDO)和非局部算子方法^[26,27](Nonlocal operator method, NOM). 受 NOM 的启发, Li 等^[28]进一步提出了键关联的弱形式 PDDO, 能有效抑制零能模式. 这些非局部建模分析方法近几年被成功应用于非线性热传导^[29,30], 功能梯度材料力学行为^[31,32]和流体运动^[33,34]等问题分析.

结合 PDDO 和 NOM 这两种非局部算子方法各自的优势, Li 等^[35,36]最近提出了近场动力学算子方法(Peridynamic operator method, PDOM), 不仅可将局部微分问题直接转化为相应的非局部积分形式, 而且扩展适用于微分的乘积. 同时, 采用 PDOM 可以直接重构各种物理问题的非局部模型, 并确保消除零能模式的影响. 本文基于 PDOM 思想推导非局部各向异性模型, 将传统各向异性理论从局部形式转化为非局部形式. 同时, 利用变分原理并结合自由振动方程, 构建了各向异性自由振动的非局部求解方案. 通过对各向异性矩形薄板、含裂缝板和含孔

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板的自由振动分析,验证了本文所重构模型和求解方法在分析含缺陷以及不连续各向异性自由振动问题方面的适用性和计算精度。

1 PDOM 理论简述

考虑二维空间 $\Omega = \mathbb{R}^2$ 中的向量函数 $\mathbf{u}(\mathbf{x})$, 如图 1 所示。

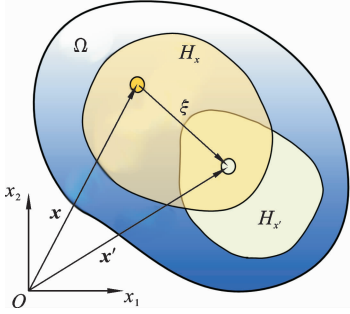


图 1 PDOM 物质点相互作用域

Fig. 1 Interaction domain of PDOM points

由一阶泰勒展开,可得:

$$\begin{cases} \eta_i(\mathbf{x}, \boldsymbol{\xi}) = u_i(\mathbf{x}') - u_i(\mathbf{x}) = \nabla u_i \cdot \boldsymbol{\xi} \\ \forall \mathbf{x} \in \Omega, \mathbf{x}' = \mathbf{x} + \boldsymbol{\xi} \in H_x \end{cases} \quad (1)$$

其中, H_x 表示点 \mathbf{x} 的相互作用域. 将式(1)在 H_x 上积分,并考虑权函数 $\bar{w}(|\boldsymbol{\xi}|)$ 以及相对位置 $\boldsymbol{\xi}^T$,得到:

$$\int_{H_x} \bar{w}(|\boldsymbol{\xi}|) \eta_i(\mathbf{x}, \boldsymbol{\xi}) \boldsymbol{\xi}^T dA_{x'} = \nabla u_i \cdot \mathbf{A} \quad (2)$$

其中,

$$\mathbf{A} = \int_{H_x} \bar{w}(|\boldsymbol{\xi}|) \boldsymbol{\xi} \otimes \boldsymbol{\xi} dA_{x'} \quad (3)$$

从式(2)得到 ∇u_i 的非局部形式:

$$\nabla u_i = \int_{H_x} \eta_i(\mathbf{x}, \boldsymbol{\xi}) \mathbf{g} dA_{x'} \quad (4)$$

其中,

$$\mathbf{g} = \bar{w}(|\boldsymbol{\xi}|) \mathbf{A}^{-1} \boldsymbol{\xi} \quad (5)$$

若使用二阶泰勒展开,则可得:

$$\eta_i \eta_j = (u_i \tilde{\nabla} u_j) \cdot \tilde{\boldsymbol{\xi}} \quad (6)$$

其中,

$$\begin{cases} \tilde{\nabla} = [\nabla_1 \curvearrowright \nabla_1 & \nabla_2 \curvearrowright \nabla_2 & \nabla_1 \curvearrowright \nabla_2 + \nabla_2 \curvearrowright \nabla_1]^T \\ \tilde{\boldsymbol{\xi}} = [\xi_1^2 & \xi_2^2 & \xi_1 \xi_2]^T \end{cases} \quad (7)$$

其中, $\tilde{\nabla}$ 是自定义的微分算子向量, \curvearrowright 是自定义的微

分内积算子,满足性质:

$$\begin{cases} u_k (\nabla_i \curvearrowright \nabla_j) u_l = (u_k \nabla_i) (\nabla_j u_l) = u_{k,i} u_{l,j} \\ u_k (a \nabla_i \curvearrowright \nabla_j b) u_l = a b u_{k,i} u_{l,j} \quad (a, b \in \mathbb{R}) \end{cases} \quad (8)$$

将式(6)在 H_x 上积分,并考虑权函数 $\tilde{w}(|\boldsymbol{\xi}|)$ 以及相对位置矢量 $\tilde{\boldsymbol{\xi}}^T$,得到:

$$\int_{H_x} \tilde{w}(|\boldsymbol{\xi}|) \eta_i \eta_j \tilde{\boldsymbol{\xi}}^T dA_{x'} = (u_i \tilde{\nabla} u_j) \cdot \tilde{\mathbf{A}} \quad (9)$$

其中,

$$\tilde{\mathbf{A}} = \int_{H_x} \tilde{w}(|\boldsymbol{\xi}|) \tilde{\boldsymbol{\xi}} \otimes \tilde{\boldsymbol{\xi}} dA_{x'} \quad (10)$$

则可从式(9)得到 $u_i \tilde{\nabla} u_j$ 的非局部形式:

$$u_i \tilde{\nabla} u_j = \int_{H_x} \eta_i \eta_j \tilde{\mathbf{g}} dA_{x'} \quad (11)$$

其中,

$$\tilde{\mathbf{g}} = \tilde{w}(|\boldsymbol{\xi}|) \tilde{\mathbf{A}}^{-1} \tilde{\boldsymbol{\xi}} \quad (12)$$

2 PDOM 各向异性模型

2.1 应变能密度

设二维空间中的各向异性材料,在整体坐标系 $x_1 O x_2$ 和材料坐标系 $1 O' 2$ 之间存在夹角 φ ,如图 2 所示。

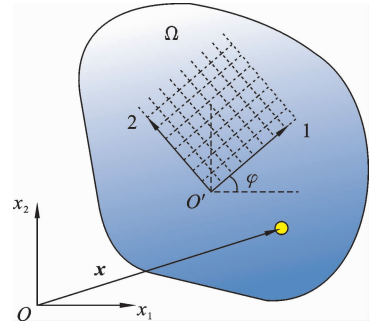


图 2 整体坐标系和材料坐标系

Fig. 2 Global and material coordinate systems

在材料坐标系中,应力-应变关系可表示为:

$$\boldsymbol{\sigma}' = \mathbf{C} \boldsymbol{\varepsilon}' \quad (13)$$

其中, $\boldsymbol{\sigma}' = [\sigma'_1 \ \sigma'_2 \ \tau'_{12}]^T$, $\boldsymbol{\varepsilon}' = [\varepsilon'_1 \ \varepsilon'_2 \ \gamma'_{12}]^T$ 分别为应力和应变, \mathbf{C} 为各向异性弹性系数矩阵,表示为:

$$\mathbf{C} = \begin{bmatrix} C_{11} & C_{12} & C_{16} \\ & C_{22} & C_{26} \\ \text{sym} & & C_{66} \end{bmatrix} \quad (14)$$

在整体坐标系中,应力 $\boldsymbol{\sigma} = [\sigma_1 \ \sigma_2 \ \tau_{12}]^T$,应变 $\boldsymbol{\varepsilon} = [\varepsilon_1 \ \varepsilon_2 \ \gamma_{12}]^T$,可分别表示为:

$$\boldsymbol{\sigma} = \mathbf{R}_m^{-1} \boldsymbol{\sigma}', \quad \boldsymbol{\varepsilon} = \mathbf{R}_m^T \boldsymbol{\varepsilon}' \quad (15)$$

其中, \mathbf{R}_m 为坐标转换矩阵,可表示为:

$$\mathbf{R}_m = \begin{bmatrix} \cos^2 \varphi & \sin^2 \varphi & 2\sin\varphi\cos\varphi \\ \sin^2 \varphi & \cos^2 \varphi & -2\sin\varphi\cos\varphi \\ -\sin\varphi\cos\varphi & \sin\varphi\cos\varphi & \cos^2 \varphi - \sin^2 \varphi \end{bmatrix} \quad (16)$$

将式(15)代入式(13),可得到:

$$\boldsymbol{\sigma} = \bar{\mathbf{C}} \boldsymbol{\varepsilon} \quad (17)$$

其中, $\bar{\mathbf{C}}$ 为整体坐标系下的各向异性弹性系数矩阵,可表示为:

$$\bar{\mathbf{C}} = \mathbf{R}_m^{-1} \mathbf{C} \mathbf{R}_m^T = \begin{bmatrix} \bar{C}_{11} & \bar{C}_{12} & \bar{C}_{16} \\ & \bar{C}_{22} & \bar{C}_{26} \\ \text{sym} & & \bar{C}_{66} \end{bmatrix} \quad (18)$$

应变能密度 W 可表示为:

$$W = \frac{1}{2} \boldsymbol{\sigma}^T \boldsymbol{\varepsilon} \quad (19)$$

应变-位移关系为 $\boldsymbol{\varepsilon} = \mathbf{L} \mathbf{u}$, $\mathbf{u} = [u_1 \ u_2]^T$ 为位移, \mathbf{L} 为微分算子矩阵,可表示为:

$$\mathbf{L} = \begin{bmatrix} \nabla_1 & 0 \\ 0 & \nabla_2 \\ \nabla_2 & \nabla_1 \end{bmatrix} \quad (21)$$

应变能密度可扩展为:

$$W = \frac{1}{2} \mathbf{u}^T (\mathbf{L}^T \bar{\mathbf{C}} \mathbf{L}) \mathbf{u} = \mathbf{u}^T \mathbf{D} \mathbf{u} \quad (21)$$

其中, \mathbf{D} 为对称微分算子矩阵,其组分可表示为:

$$D_{ij} = \bar{\mathbf{C}} : (\mathbf{L} \cdot_i \tilde{\otimes} \mathbf{L} \cdot_j) \quad (22)$$

其中 $\mathbf{L} \cdot_i$ 表示矩阵 \mathbf{L} 的第 i 列全部元素, $\tilde{\otimes}$ 是自定义针对微分的张量积算子,则矩阵 \mathbf{D} 的上三角元素可表示为:

$$\begin{cases} D_{11} = [\bar{C}_{11} \ \bar{C}_{66} \ \bar{C}_{16}] \tilde{\nabla} \\ D_{12} = [\bar{C}_{16} \ \bar{C}_{26} \ \bar{C}_{66}] \tilde{\nabla} + (\bar{C}_{12} - \bar{C}_{66}) \nabla_1 \rightsquigarrow \nabla_2 \\ D_{21} = [\bar{C}_{16} \ \bar{C}_{26} \ \bar{C}_{66}] \tilde{\nabla} + (\bar{C}_{12} - \bar{C}_{66}) \nabla_2 \rightsquigarrow \nabla_1 \\ D_{22} = [C_{66} \ C_{22} \ C_{26}] \tilde{\nabla} \end{cases} \quad (23)$$

将式(22)代入式(21),可得到:

$$W = \frac{1}{2} [(\bar{C}_{12} - \bar{C}_{66}) \theta^2 + \Theta] \quad (24)$$

其中 θ 为体应变,可表示为:

$$\theta = u_{i,i} = \nabla \cdot \mathbf{u} \quad (25)$$

$$\Theta = \mathbf{u}^T \tilde{\mathbf{D}} \mathbf{u} \quad (26)$$

其中

$$\begin{cases} \tilde{\mathbf{D}} = \begin{bmatrix} \tilde{D}_{11} & \tilde{D}_{12} \\ \tilde{D}_{12} & \tilde{D}_{22} \end{bmatrix}, \quad [\tilde{D}_{11} \ \tilde{D}_{22} \ \tilde{D}_{12}]^T = \hat{\mathbf{C}} \tilde{\nabla} \\ \hat{\mathbf{C}} = \begin{bmatrix} \bar{C}_{11} - \bar{C}_{12} + \bar{C}_{66} & \bar{C}_{66} & \bar{C}_{16} \\ & \bar{C}_{22} - \bar{C}_{12} + \bar{C}_{66} & \bar{C}_{26} \\ \text{sym} & & \bar{C}_{66} \end{bmatrix} \end{cases} \quad (27)$$

将式(4)代入式(25),可得:

$$\theta = \int_{H_x} \mathbf{g}^T \boldsymbol{\eta} dA_x \quad (28)$$

将式(11)代入式(26)、式(27),可得:

$$\Theta = \int_{H_x} \boldsymbol{\eta}^T \tilde{\mathbf{D}} \boldsymbol{\eta} dA_x \quad (29)$$

其中,

$$[\bar{D}_{11} \ \bar{D}_{22} \ \bar{D}_{12}]^T = \hat{\mathbf{C}} \tilde{\mathbf{g}} \quad (30)$$

各向异性材料的应变能密度的非局部形式,由式(5)、(12)、(24)、(28)-(30)构成.

2.2 求解方案

将式(28)和式(29)离散为:

$$\begin{cases} \theta_k = \sum_{x_j \in H_k} \mathbf{g}_{jk}^T \boldsymbol{\eta}_{jk} A_j \\ \Theta_k = \sum_{x_j \in H_k} \boldsymbol{\eta}_{jk}^T \tilde{\mathbf{D}}_{jk} \boldsymbol{\eta}_{jk} A_j \end{cases} \quad (31)$$

其中, k 表示 Ω 中第 k 个点, $\boldsymbol{\eta}_{jk} = \mathbf{u}_j - \mathbf{u}_k$. 将式(31)写成矩阵形式:

$$\begin{cases} \theta_k = \mathbf{B}_k \bar{\mathbf{u}}_k \\ \Theta_k = \bar{\mathbf{u}}_k^T \hat{\mathbf{D}}_k \bar{\mathbf{u}}_k \end{cases} \quad (32)$$

式中

$$\begin{cases} \bar{\mathbf{u}}_k = [\mathbf{u}_{j_1}^T \ \cdots \ \mathbf{u}_{j_{N_k}}^T \ \mathbf{u}_k^T]^T \\ \mathbf{B}_k = \left[\mathbf{g}_{j_1 k}^T A_{j_1} \ \cdots \ \mathbf{g}_{j_{N_k} k}^T \tilde{\mathbf{L}}_{j_{N_k} k} A_{j_{N_k}} \ - \sum_{i=1}^{N_k} \mathbf{g}_{j_i k}^T A_{j_i} \right] \\ \hat{\mathbf{D}}_k = \begin{bmatrix} \bar{\mathbf{D}}_{j_1 k} A_{j_1} & \cdots & \mathbf{0} & -\bar{\mathbf{D}}_{j_1 k} A_{j_1} \\ \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \cdots & \bar{\mathbf{D}}_{j_{N_k} k} A_{j_{N_k}} & -\bar{\mathbf{D}}_{j_{N_k} k} A_{j_{N_k}} \\ -\bar{\mathbf{D}}_{j_1 k} A_{j_1} & \cdots & -\bar{\mathbf{D}}_{j_{N_k} k} A_{j_{N_k}} & \sum_{i=1}^{N_k} \bar{\mathbf{D}}_{j_i k} A_{j_i} \end{bmatrix} \end{cases} \quad (33)$$

其中 N_k 表示 H_k 中点的个数, $j_i k$ 中的 j_i 表示 H_k 中第 i 个物质点的全局索引. 将式 (32) 代入式 (24), 可得:

$$W_k = \frac{1}{2} \bar{\mathbf{u}}_k^T \mathbf{K}_k \bar{\mathbf{u}}_k \quad (34)$$

其中,

$$\mathbf{K}_k = (\bar{C}_{12} - \bar{C}_{66}) \mathbf{B}_k^T \mathbf{B}_k + \hat{\mathbf{D}}_k \quad (35)$$

对于静力学问题, 能量泛函可通过最小二乘原理表示为:

$$\Pi(\mathbf{u}) = \int_{\Omega} W dA - \int_{\Omega} \mathbf{u}^T \mathbf{b} dA - \int_{\partial\Omega_i} \mathbf{u}^T \bar{\mathbf{t}} dS \quad (36)$$

式中, $\bar{\mathbf{t}}$ 表示边界 $\partial\Omega_i$ 上的牵引力. 将式 (34) 代入式 (36) 的离散形式, 并令 $\delta\Pi=0$, 可得:

$$\mathbf{K}\mathbf{U}=\mathbf{P} \quad (37)$$

其中,

$$\begin{cases} \mathbf{K} = \sum_{x_k \in \Omega} \mathbf{K}_k A_k \\ \mathbf{P} = \sum_{x_k \in \Omega} \mathbf{b}_k A_k + \sum_{x_k \in \partial\Omega_i} \bar{\mathbf{t}}_k S_k \\ \mathbf{U} = [\mathbf{u}_1^T \quad \mathbf{u}_2^T \quad \cdots \quad \mathbf{u}_{N_{\text{total}}}^T]^T \end{cases} \quad (38)$$

式中, \mathbf{K} 表示整体刚度矩阵, S_k 表示边界物质点 x_k 的面积, N_{total} 表示 Ω 中总物质点个数. 此处求和符号与有限元方法中的装配运算一致.

不考虑阻尼影响的自由振动方程的特征值形式可表示为:

$$\mathbf{K}\boldsymbol{\phi} - \omega^2 \mathbf{M}\boldsymbol{\phi} = \mathbf{0} \quad (39)$$

求解式 (39), 可确定固有圆频率 ω 和固有振型 $\boldsymbol{\phi}$, \mathbf{M} 为物质点质量矩阵, 表示为:

$$\mathbf{M} = \begin{bmatrix} \rho A_1 & 0 & 0 & 0 & 0 \\ 0 & \rho A_2 & 0 & 0 & 0 \\ 0 & 0 & \rho A_3 & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & \rho A_{N_{\text{total}}} \end{bmatrix} \quad (40)$$

其中, ρ 为材料密度. 式 (39) 可写为标准特征值形式:

$$\mathbf{H}\boldsymbol{\phi} = \lambda\boldsymbol{\phi} \quad (41)$$

其中, $\mathbf{H} = \mathbf{M}^{-1} \mathbf{K}$, $\lambda = \omega^2$. 数值算法流程图如图 3 所示.

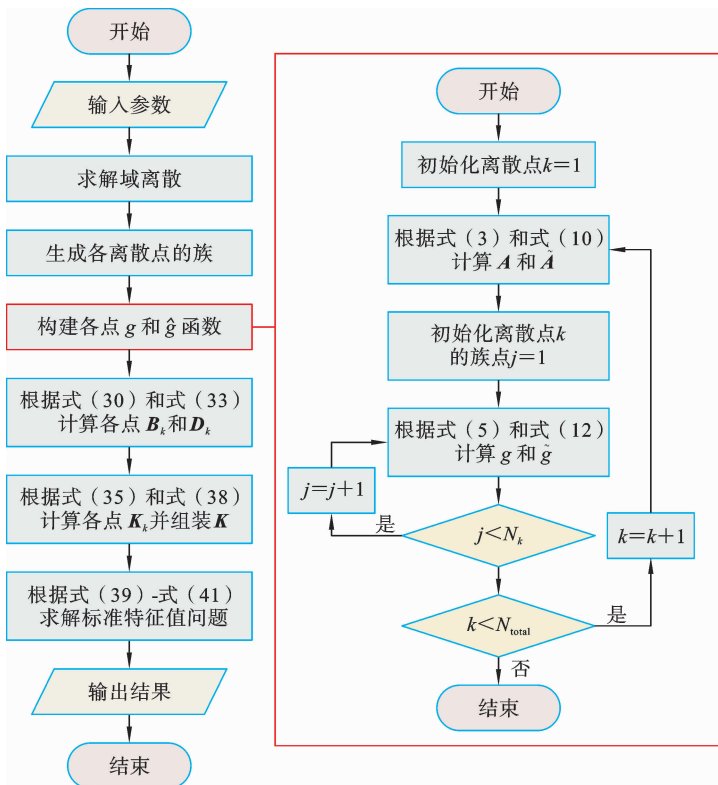


图 3 数值算法流程图

Fig. 3 Flowchart of numerical algorithm

3 算例验证

本节应用基于 PDOM 方法重构的非局部各向异性模型开展自由振动分析. 首先通过求解各向异性矩形薄板的自振频率, 验证本文方法的准确性和收敛性. 其次, 通过求解各向异性矩形含裂缝板和含孔板在不同各向异性偏角下的各阶自振频率, 验证本文方法计算各向异性含缺陷结构自由振动问题的精确性, 且通过模态分析, 验证本文方法计算结果的连续性与稳定性. 算例中弹性刚度系数均取 $C_{11} = 15.32 \text{ GPa}$, $C_{22} = 5.11 \text{ GPa}$, $C_{66} = 4.8 \text{ GPa}$, $C_{12} = 1.28 \text{ GPa}$, $C_{16} = 0.16 \text{ GPa}$, $C_{26} = 0.11 \text{ GPa}$. 材料密度 $\rho = 2500 \text{ kg/m}^3$. 自振频率的表达式为:

$$f = \frac{\omega}{2\pi} \quad (42)$$

3.1 各向异性矩形薄板

矩形薄板尺寸为 $L = 0.4 \text{ m}$, $W = 0.2 \text{ m}$, 左端固定. 整体坐标系和材料坐标系之间的夹角为 $\varphi = \pi/2$. PDOM 数值计算中, 将矩形板均匀离散为 5000 个物质点. 离散间距 $\Delta x = 4 \text{ mm}$, 近场范围尺寸取 $\delta = 3\Delta x$. 为进行对比, 同时进行有限元计算, 矩形板被离散为 11486 个三角形单元, 单元边长为 4 mm .

表 1 中给出了各向异性矩形薄板前 8 阶自振频率, 并与有限元解进行对比. 从表中可以看出, PDOM 解与有限元解非常一致, 最大相对误差为 0.881% .

表 1 矩形板自振频率 ($\varphi = \pi/2$)

Table 1 Frequencies of the rectangular plate ($\varphi = \pi/2$)

阶次	f_{PDOM} (Hz)	f_{FEM} (Hz)	相对误差 (%)
1	259.43	260.24	0.311
2	892.41	884.85	0.854
3	1141.4	1136.8	0.405
4	2473.9	2460.0	0.565
5	2674.1	2650.9	0.875
6	3558.9	3545.1	0.389
7	4383.6	4369.5	0.323
8	4431.1	4392.4	0.881

分别取不同的离散间距 $\Delta x = 4 \text{ mm}$, 6 mm , 8 mm , 10 mm , 图 4 为前三阶自振频率的相对误差. 从图中可以看出, 相对误差随着物质点离散间距的减小而快速减小, 说明本方法具有较好的收敛性.

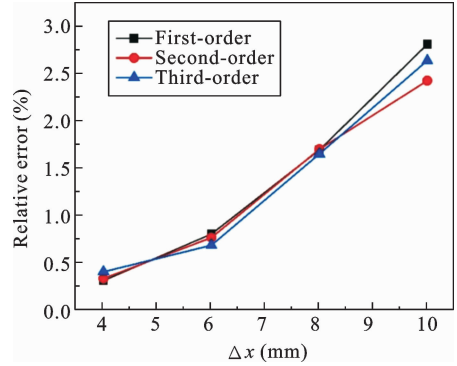


图 4 不同离散间距下前三阶频率的相对误差

Fig. 4 Relative error of first 3 frequencies for different discrete distance

3.2 各向异性矩形含裂缝板

考虑各向异性矩形含裂缝板的自由振动, 几何条件如图 5 所示. $L = 0.4 \text{ m}$, $W = 0.2 \text{ m}$, 板中心裂缝长度 $m = 0.2 \text{ m}$, 宽度 $n = 4 \text{ mm}$, 左端固定. PDOM 数值计算中, 将含裂缝矩形板均匀离散为 4900 个物质点. 离散间距 $\Delta x = 4 \text{ mm}$, 近场范围尺寸取 $\delta = 3\Delta x$. 有限元计算中, 含裂缝矩形板被离散为 10912 个三角形单元, 单元边长为 4 mm .

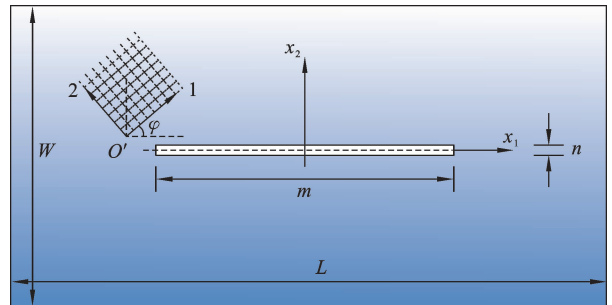


图 5 各向异性矩形含裂缝板几何条件

Fig. 5 Geometry conditions of anisotropic rectangular cracked plate

表 2 给出了含裂缝板在偏角 $\varphi = 0, \pi/6, \pi/3, \pi/2$ 下前三阶自振频率的 PDOM 和 FEM 计算结果. 从表中可以看出, 前三阶自振频率最大相对误差

表 2 各向异性矩形含裂缝板自振频率

Table 2 Natural frequencies of the anisotropic rectangular cracked plate

阶数	φ	f_{PDOM} (Hz)	f_{FEM} (Hz)	相对误差(%)
1 阶	0	368.53	371.92	0.911
	$\pi/6$	333.18	336.61	1.019
	$\pi/3$	269.06	268.02	0.388
	$\pi/2$	234.40	236.97	1.085
2 阶	0	1476.4	1476.7	0.020
	$\pi/6$	1277.0	1272.7	0.338
	$\pi/3$	981.59	984.72	0.318
	$\pi/2$	882.74	884.36	0.183
3 阶	0	1493.2	1508.8	1.033
	$\pi/6$	1418.8	1427.5	0.609
	$\pi/3$	1188.3	1180.9	0.627
	$\pi/2$	1048.9	1048.4	0.048

为 1.085%，本文方法所得结果与 FEM 高度吻合。图 6 给出了各向异性偏角 $\varphi = \pi/3$ 时 PDOM 与 FEM 前 3 阶全位移模态。可以看出，前 3 阶 PDOM 与 FEM 的位移模态结果吻合，模态位移场规则、连续。

3.3 各向异性矩形含孔板

考虑各向异性矩形含孔板的自由振动，几何条件如图 7 所示。 $L=0.4\text{ m}$ ， $W=0.2\text{ m}$ ，中心圆孔半径为 $a=0.04\text{ m}$ ，左端固定。 r 和 ϕ 分别为相应极坐标系中的极径与极角。PDOM 数值计算中，将含孔矩形板均匀离散为 4668 个物质点，离散间距 $\Delta x=4\text{ mm}$ ，近场范围尺寸取 $\delta=3\Delta x$ 。有限元计算中，含孔矩形板被离散为 10566 个三角形单元，单元边长为 4 mm 。

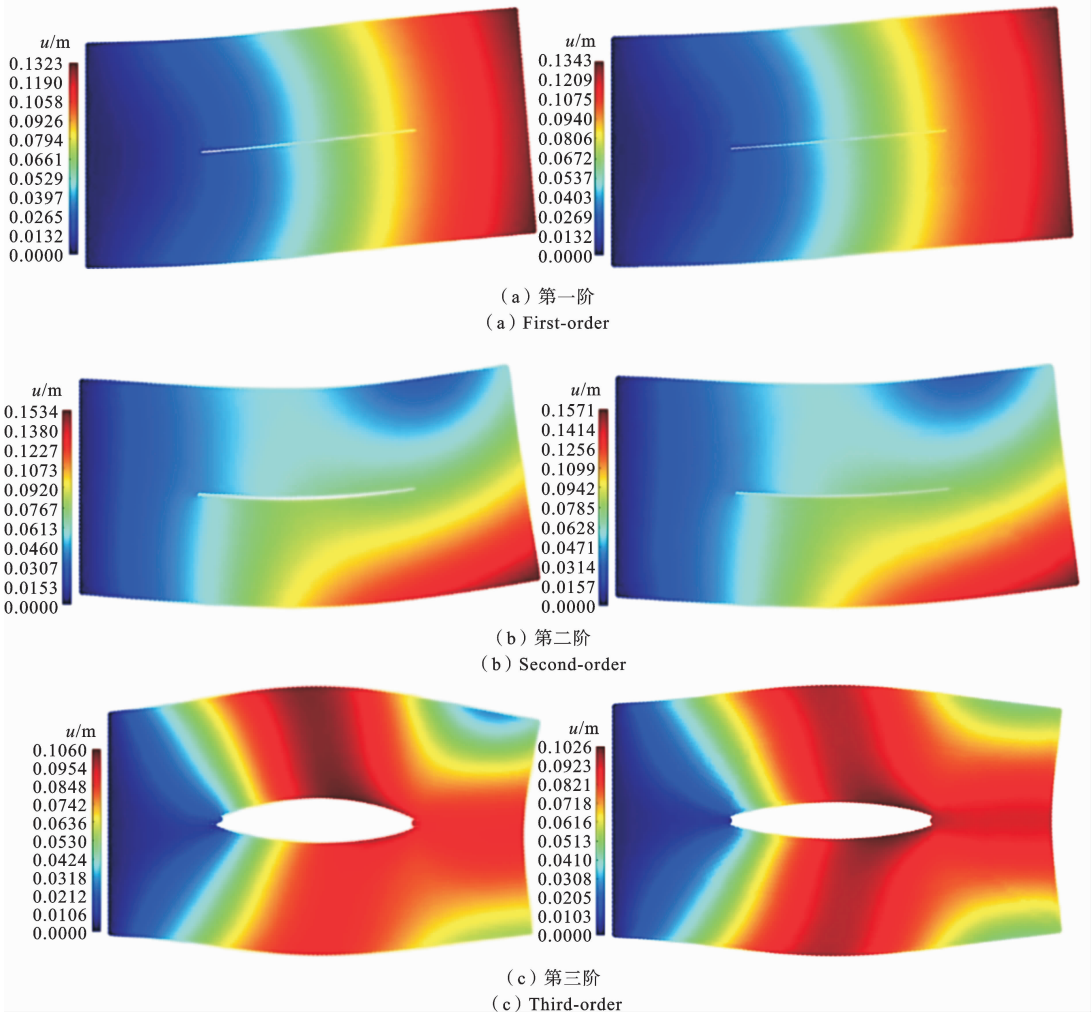


图 6 各向异性偏角 $\varphi = \pi/3$ 时含裂缝板前 3 阶全位移模态(左栏为 PDOM 结果,右栏为 FEM 结果)

Fig. 6 First 3 order displacement modes of the anisotropic cracked plate under off-angle $\varphi = \pi/3$ (PDOM results on the left, FEM results on the right)

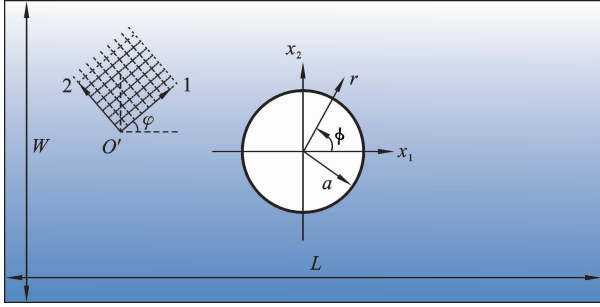


图 7 各向异性矩形含孔板几何条件

Fig. 7 Geometry conditions of anisotropic rectangular plate with a hole

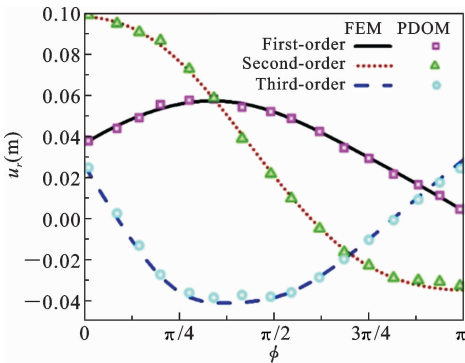
表 3 给出了含孔板在偏角 $\varphi = 0, \pi/6, \pi/3, \pi/2$ 下前 3 阶自振频率的 PDOM 和 FEM 计算结果. 从表中可以看出, 前 3 阶自振频率最大相对误差为 0.696%, 本文方法所得结果与 FEM 高度吻合. 图 8 给出了含孔板的前 3 阶位移模态在 $r = 0.07 \text{ m}$, $0 \leq \phi \leq \pi$ 处分布的 FEM 与 PDOM 数值结果对比. 图 9 给出了含孔板在不同偏角下的第 3 阶位移模态在 $r = 0.07 \text{ m}$, $0 \leq \phi \leq \pi$ 处分布的 FEM 与 PDOM 数值结果对比. 可以看出, 在偏角 $\varphi = \pi/2$ 情况下, 前 3 阶径向位移模态 u_r 和环向位移模态 u_ϕ 的 PDOM 与 FEM 结果基本一致; 在偏角 $\varphi = 0, \pi/6, \pi/3, \pi/2$ 下, 第 3 阶径向位移模态 u_r 和环向位移模态 u_ϕ 的 PDOM 与 FEM 结果吻合. 图 10 给出了偏角 $\varphi = \pi/6$

时 PDOM 与 FEM 前 3 阶全位移模态. 可以看出, 前 3 阶 PDOM 与 FEM 的位移模态结果基本一致, 且模态位移场规则、连续、稳定, 没有出现明显的数值振荡.

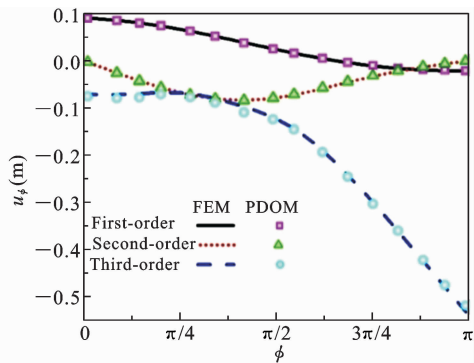
表 3 各向异性矩形含孔板自振频率

Table 3 Natural frequencies of the anisotropic rectangular plate with a hole

阶数	φ	f_{PDOM} (Hz)	f_{FEM} (Hz)	相对误差 (%)
1 阶	0	387.88	387.03	0.220
	$\pi/6$	352.15	352.69	0.153
	$\pi/3$	283.87	285.10	0.431
	$\pi/2$	251.29	253.05	0.696
2 阶	0	1377.3	1379.4	0.152
	$\pi/6$	1238.1	1236.6	0.121
	$\pi/3$	942.66	940.47	0.233
	$\pi/2$	834.08	832.01	0.249
3 阶	0	1591.6	1584.0	0.480
	$\pi/6$	1459.2	1455.5	0.254
	$\pi/3$	1263.0	1262.2	0.063
	$\pi/2$	1160.0	1161.2	0.103



(a) 径向位移模态 u_r
(a) Radial displacement modes u_r



(b) 环向位移模态 u_ϕ
(b) Circular displacement modes u_ϕ

图 8 FEM 和 PDOM 求得的前 3 阶位移模态在 $r = 0.07 \text{ m}$ 处分布 ($\varphi = \pi/2$)

Fig. 8 First 3 order displacement modes obtained with FEM and PDOM at $r = 0.07 \text{ m}$ ($\varphi = \pi/2$)

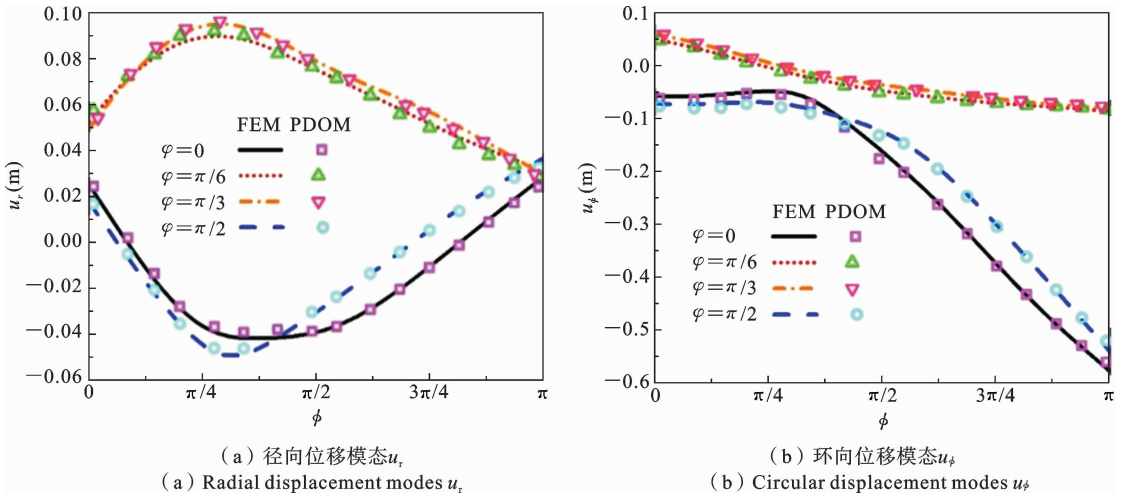


图 9 不同各向异性偏角 φ 下 FEM 和 PDOM 求得的第 3 阶位移模式在 $r=0.07$ m 处分布

Fig. 9 Third-order displacement modes obtained with FEM and PDOM at $r=0.07$ m under different anisotropic off-angle φ

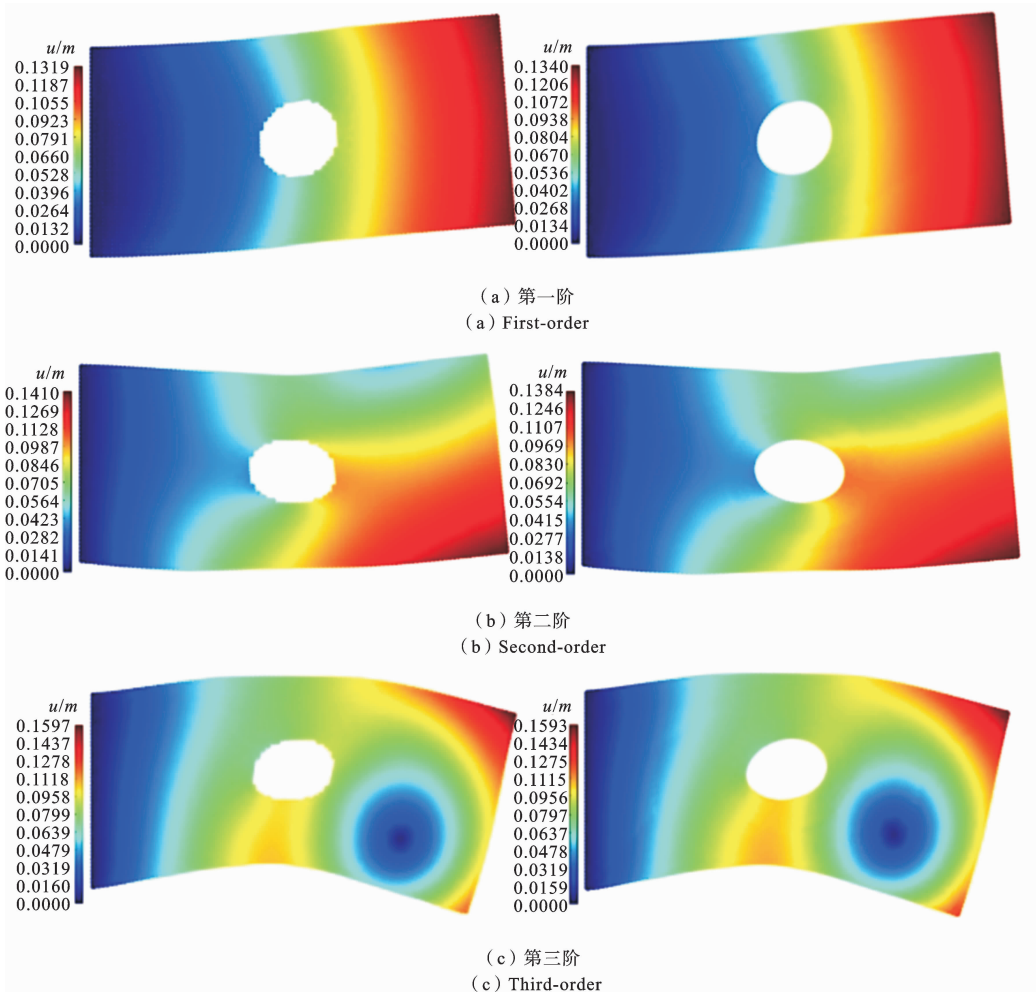


图 10 各向异性偏角 $\varphi=\pi/6$ 时含孔板前 3 阶全位移模式(左栏为 PDOM 结果,右栏为 FEM 结果)

Fig. 10 First 3 order displacement modes of the anisotropic plate with a hole under off-angle $\varphi=\pi/6$ (PDOM results on the left, FEM results on the right)

4 结论

基于近场动力学算子方法,本文提出了一种新颖的非局部各向异性模型,并应用于各向异性板自由振动问题的研究.通过结合 PDOM 的特性,即将局部微分及其乘积转化为非局部积分,本文将经典各向异性理论中关于应变能密度的局部表达重新构建为非局部形式.同时,借助变分原理并引入自由振动方程,构建了适用于解决各向异性自由振动问题的 PDOM 数值求解方案.通过计算各向异性矩形薄板的自由振动问题,证明了该方法具备较好的准确性和收敛性.通过计算各向异性矩形含裂缝板和各向异性矩形含孔板的自由振动问题,将所提模型的计算结果与有限元分析结果进行了对比.结果表明,本文提出的模型在处理含缺陷、不连续各向异性板自由振动问题时表现出色,具备了较好的稳定性和连续性,且能够获得高度精确的计算结果.

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Free Vibration Analysis for Anisotropic Plates Based on Peridynamic Operator Method

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Abstract Anisotropic materials find widespread applications across various engineering domains. The investigation on vibrational properties of anisotropic materials holds significance for structural vibration mitigation and safety design. This paper introduces a novel approach, the peridynamic operator method (PDOM), to construct a non-local anisotropic model and applies it to the analysis of free vibrations in anisotropic plates. The model incorporates the unique feature of PDOM, which transforms local differentials and their products into non-local integrals, thereby reformulating the strain energy density from its local form to a non-local form within classical anisotropic theory. Additionally, the paper employs a variational principle and introduces the free vibration equation to develop a PDOM solution for anisotropic free vibration problems. Three numerical examples are provided, including the free vibrations of a thin anisotropic rectangular plate, an anisotropic rectangular plate with cracks, and an anisotropic rectangular plate with holes. The results are compared with finite element results, showcasing the model's convergence, stability, and high computational accuracy in dealing with free vibrations of anisotropic plates with defects and discontinuities.

Key words free vibration, anisotropic, PDOM, non-local, variational principle