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三维弹性体夹芯的夹芯板振动计算方法研究

张彤彤，王纬波，刘媛慧，吴　健，吴文伟
（中国船舶科学研究中心 船舶振动噪声重点实验室，江苏 无锡 214082）

摘要：目前夹芯结构由于其轻质、减振、隔声等优良性能，已在隔声壁板、减振结构设计等领域得到广泛的应用。夹芯板在低频振动中通常易于表现为弯曲振动模式，但夹芯板面板材料和芯材材料的性能往往差别较大，在中高频段易产生较为丰富的振动模式，因此建立一种适用于各个频段的夹芯板振动理论预报方法是十分必要的。本文发展了一种新的夹芯板振动理论计算方法，对夹芯板的芯材应用三维弹性力学理论，面板应用薄板理论。本文建立的理论方法可综合体现夹芯板弯曲变形模式和膨胀变形模式，容许夹芯板产生法向形变，可显著提高中高频夹芯板的振动预报计算精度。通过算例将本文理论方法的计算结果与其它理论计算方法和有限元计算的结果进行了对比，证明了本文理论方法的有效性和优势。

关键词：夹芯板；振动；弹性力学
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Vibration calculation method for sandwich panels with three-dimensional elastomer core

ZHANG Tong-tong, WANG Wei-bo, LIU Yuan-hui, WU Jian, WU Wen-wei
(National Key Laboratory on Ship Vibration & Noise, China Ship Scientific Research Center, Wuxi 214082, China)

Abstract: Sandwich structures have been widely used in fields of sound insulation panels and vibration reduction structure design due to their lightweight, excellent performance in vibration reduction and sound insulation. Usually, sandwich panels in vibration tend to exhibit bending vibration modes in the low frequency band. However, the properties of sandwich skin materials and core materials often differ significantly, resulting in rich vibration modes in the midium and high frequency bands. Therefore, it is necessary to establish a theoretical prediction method for sandwich panel vibration applicable to various frequency bands. In this paper, a new theoretical calculation method for sandwich panel vibration was developed, in which three-dimensional elastic mechanics theory was applied to core material of sandwich panel while thin plate theory was applied to skin material. The theoretical method presented in this paper can reflect the bending deformation mode and dilatational deformation mode of sandwich panels, and allow for normal deformation of sandwich panels. As a result, the method can significantly improve the accuracy of vibration prediction calculation for medium and high frequency bands. Comparison of the calculation results of the theoretical method in this paper with those of other theoretical calculation methods and finite element method by an example proved the effectiveness and advantages of the theoretical method in this paper.

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作者简介：张彤彤(1990-)，女，高级工程师；吴文伟(1969-)，男，研究员，通讯作者，E-mail: wuww@cssrc.com.cn。

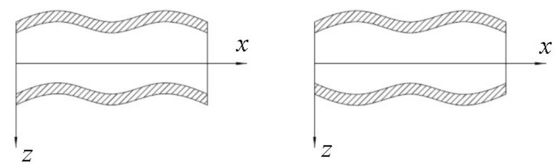
Key words: sandwich panel; vibration; elastic mechanics

0 引 言

夹芯结构是复合层结构的一种,它是由两层高强度的薄表层面板和通常软而轻的芯材所组成。夹芯结构具有重量轻、强度高和刚性大等特点,若适当选择面板和芯材,还可获得良好的减振、隔声、隔热等优良性能。由于这些优点,夹芯结构在军用和民用领域得到越来越广泛的应用。

针对夹芯板,上世纪学者们已经开展了丰富的静力学、振动、抗冲击等方面的研究,针对夹芯板在外载荷作用下变形响应的特点,提出了多种理论模型。Reissner[1]在1948年提出的夹芯板理论,将夹芯板面板看作一层薄膜,认为面板只承受面内力而忽略其本身的抗弯刚度,芯材则被认为只承受剪力;Hoff[2]在Reissner工作的基础上,进一步将面板视为薄板,计及面板的抗弯刚度,此理论可解决Reissner夹芯板理论不能解决的多种问题,如集中载荷作用下板弯曲问题、固支边界板弯曲时边界附近应力集中的问题等;1965年,Mead[3]进一步考虑了面板和芯材之间的法向作用力,并用复模量的方式考虑了粘弹性芯材阻尼的影响;之后,Mead等[4-5]又针对粘弹性阻尼夹芯梁和夹芯板展开了一系列的研究,研究了模态损耗因子对波长和芯材力学性能的依赖性,求解了不同边界条件下带加强筋的阻尼夹芯板的振动特性;Kurtze等[6]以无限大夹芯板为对象,研究了夹芯板的振动特征随频率变化的规律,为夹芯板振动分频段研究提供了指导思路。

以上的研究针对的均是夹芯板弯曲变形模式(如图1(a)所示),未考虑其膨胀变形模式(如图1(b)所示)的影响。为将膨胀变形模式的影响纳入计算,Ford等[7]首先指出在之前的研究中忽视了芯层胀缩波(dilatational wave)共振效应的影响;之后Smolenski等[8]用能量法分析了夹芯板弯曲变形模式和膨胀变形模式的特征频率,并用试验验证了膨胀变形模式对夹芯板隔声性能的影响;Dym、Spilios等[9-10]针对二维无限大夹芯板,将芯材的位移函数在一阶剪切变形理论的基础上,增加一个表征芯材膨胀变形模式的变量 ,从而可分别计算两种变形模式,然后将两种变形模式的结果进行叠加。但这种计算方法也存在一定的缺点,即只能计算两侧面板性能和几何尺寸一致的夹芯板,对两侧面板不一致的夹芯板不适用。



(a) 弯曲变形 (b) 膨胀变形

图1 夹芯板弯曲变形和膨胀变形模式

Fig.1 Bending and dilatationadeformation modes of sandwich panel

另一个对夹芯板振动进行求解的方法路线是抛弃关于应力和位移的人为假设,对夹芯板的面板和芯材均采用三维弹性力学方法,如Vlasov[11]提出的初始函数法,范家让、胡文峰等[12-13]发展的基于状态空间法的多层板理论等。用三维弹性力学方法得到的夹芯板计算结果为精确解,包含了夹芯板各种振动模式,可大大提高计算精度,但也有计算过程复杂、夹芯板振动运动方程无简明表达式等缺点。

近年来,一些新的夹芯板计算方法被持续提出和应用。Ferreira等[14]提出了一种适用于具有层合板面板和粘弹性芯材夹芯板的分层数值模型;Huang等[15]针对弹性-粘弹性-弹性夹芯板,开发了两种使用不同层合板理论的有限元模型,并通过数值算例和试验结果验证了两种模型的有效性;Safari等[16]采用哈密顿原理获得了具有粘弹性横向柔性芯材的夹芯板的振动运动方程,并在计算中考虑了粘弹性夹芯材料属性的频率依赖性;Biglari等[17]提出了一种基于三层混合理论的柔性芯双曲面夹芯壳静态问题和自由振动的封闭解,在理论中考虑了夹芯厚度上位移的非线性分布;田媛等[18]针对轻质波纹夹芯板的非线性动力响应进行了研究;朱锡等[19]建立了二维空间夹芯结构的声振理论模型,并通过试验验证了夹芯板在水中的声振特性;李泽成等[20]研究了内外等压载荷作用下复合材料夹芯圆柱壳的稳定性及面板和芯材之间的分层扩展行为。

总体来说，传统的夹芯板理论大多数都引入了一些人为的简化假设，如仅考虑芯材的剪切变形、忽略芯材法向挤压变形的影响、假定上下面板的挠度一致等。这些传统的理论在进行夹芯板低频振动计算时，可以取得良好的效果。但由于芯材的模量通常较小，在进行中高频振动时，振动波长较小，容易激发出丰富的振动模式。芯材的振动不再遵循传统夹芯板理论剪切弯曲振动的特点，而是涨缩波、剪切波并存，夹芯板的弯曲变形模式和膨胀变形模式均会对其振动造成明显的影响。此时若依然采用传统理论计算夹芯板振动，便会导致计算误差出现。而若对夹芯板的面板和芯材均应用三维弹性力学理论以避免计算误差，又往往会使计算过程复杂，计算效率较低。

为解决这个问题，本文发展了一种新的理论方法，令夹芯板的面板服从薄板假设，而芯材不作简化性的假设，直接应用三维弹性力学方法，这样便可包含芯材的所有振动模式，且不必将夹芯板的弯曲变形模式和膨胀变形模式分别计算再进行叠加，可进一步简化计算过程。采用本文的理论方法计算夹芯板的振动，可在振动变形计算中包含上述两种变形模式的贡献，明显提高计算精度，同时可有效提高计算效率（以1 m尺寸的夹芯板为例，本文理论方法的计算耗时约为基于状态空间法的多层板理论耗时的60%，小于有限元仿真方法耗时的1%），对中高频夹芯板振动响应的预报具有重要意义。

1 简支矩形夹芯板芯材与面板振动理论计算方法

1.1 芯材振动计算

芯材是夹芯板振动计算中的关键，但由于其丰富的振动模式，也是夹芯板振动计算中的难点。传统理论忽略芯材法向变形的人为假设是造成中高频计算误差的主要原因之一，因此，本文理论中不对芯材的振动作过多假设，而直接应用三维弹性理论。如此，芯材的涨缩效应和剪切效应可同时在芯材位移中体现出来，而不必单独对芯材膨胀变形时的法向变形进行额外处理。

本文研究对象为简支矩形夹芯板，夹芯板结构及坐标系如图2所示，夹芯板面板1的厚度为t_1，面板2的厚度为t_2，芯材厚度为h，夹芯板长度为a，宽度为b。夹芯板各部分应力如图3所示，其中面板1和芯材之间的界面称为“界面1”，面板2和芯材之间的界面称为“界面2”，在界面处，面板和芯材遵循“位移连续”的原则。

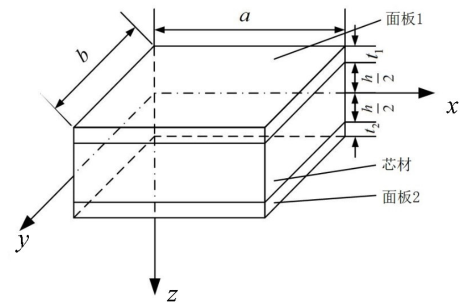


图2 夹芯板的结构

Fig.2 Structure of sandwich panel

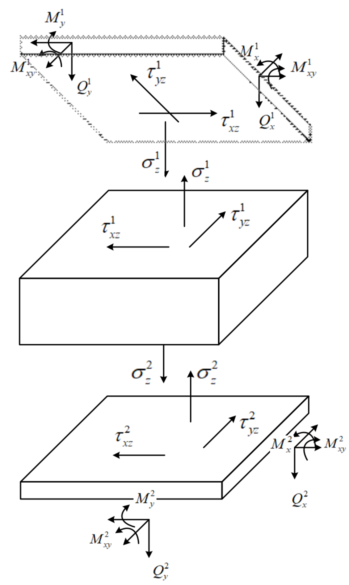


图3 夹芯板各部分应力

Fig.3 Stress of each part of sandwich panel

将芯材视为各向同性弹性体，在忽略重力等体积力的情况下，其运动微分方程[21]为

$$\begin{cases} \dfrac{\partial \sigma_x}{\partial x} + \dfrac{\partial \tau_{xy}}{\partial y} + \dfrac{\partial \tau_{xz}}{\partial z} = \rho \dfrac{\partial^2 u}{\partial t^2} \\ \dfrac{\partial \tau_{yx}}{\partial x} + \dfrac{\partial \sigma_y}{\partial y} + \dfrac{\partial \tau_{yz}}{\partial z} = \rho \dfrac{\partial^2 v}{\partial t^2} \\ \dfrac{\partial \tau_{zx}}{\partial x} + \dfrac{\partial \tau_{zy}}{\partial y} + \dfrac{\partial \sigma_z}{\partial z} = \rho \dfrac{\partial^2 w}{\partial t^2} \end{cases} \tag{1}$$

式中，u、v、w分别为芯材在x、y、z轴三个方向的位移分量，ρ为芯材的密度。

芯材的物理方程为

$$\begin{cases}\sigma_x=\lambda\theta+2G\dfrac{\partial u}{\partial x}\quad,\quad\tau_{xy}=G\left(\dfrac{\partial v}{\partial x}+\dfrac{\partial u}{\partial y}\right)\\ \sigma_y=\lambda\theta+2G\dfrac{\partial v}{\partial y}\quad,\quad\tau_{yz}=G\left(\dfrac{\partial w}{\partial y}+\dfrac{\partial v}{\partial z}\right)\\ \sigma_z=\lambda\theta+2G\dfrac{\partial w}{\partial z}\quad,\quad\tau_{xz}=G\left(\dfrac{\partial u}{\partial z}+\dfrac{\partial w}{\partial x}\right)\end{cases}\tag{2}$$

式中，$\lambda=\dfrac{2\mu G}{1-2\mu}$，$G$为芯材的剪切模量，$\mu$为芯材的泊松比，$\theta=\dfrac{\partial u}{\partial x}+\dfrac{\partial v}{\partial y}+\dfrac{\partial w}{\partial z}$。

振动中芯材的位移可表示为

$$\begin{aligned}u(x,y,z,t)&=\tilde{u}(x,y,z)\mathrm{e}^{\mathrm{i}\omega t}\\ v(x,y,z,t)&=\tilde{v}(x,y,z)\mathrm{e}^{\mathrm{i}\omega t}\\ w(x,y,z,t)&=\tilde{w}(x,y,z)\mathrm{e}^{\mathrm{i}\omega t}\\ \theta(x,y,z,t)&=\tilde{\theta}(x,y,z)\mathrm{e}^{\mathrm{i}\omega t}\end{aligned}\tag{3}$$

式中，$\tilde{\theta}=\dfrac{\partial\tilde{u}}{\partial x}+\dfrac{\partial\tilde{v}}{\partial y}+\dfrac{\partial\tilde{w}}{\partial z}$。

为表示简便，下面的推导中将忽略变量中的时间项$\mathrm{e}^{i\omega t}$。将式(2)、式(3)代入式(1)，得到

$$\begin{cases}(\lambda+G)\dfrac{\partial\tilde{\theta}}{\partial x}+\left(G\nabla^2+\rho\omega^2\right)\tilde{u}=0\\ (\lambda+G)\dfrac{\partial\tilde{\theta}}{\partial y}+\left(G\nabla^2+\rho\omega^2\right)\tilde{v}=0\\ (\lambda+G)\dfrac{\partial\tilde{\theta}}{\partial z}+\left(G\nabla^2+\rho\omega^2\right)\tilde{w}=0\end{cases}\tag{4}$$

式中，$\nabla^2=\dfrac{\partial^2}{\partial x^2}+\dfrac{\partial^2}{\partial y^2}+\dfrac{\partial^2}{\partial z^2}$。

采用文献[22]提出的位移函数法，引入2个位移函数F和Φ，将位移表示为位移函数的微分形式，

$$\begin{cases}\tilde{u}=-\dfrac{\partial^2F}{\partial x\partial z}-\dfrac{\partial\Phi}{\partial y}\\ \tilde{v}=-\dfrac{\partial^2F}{\partial y\partial z}+\dfrac{\partial\Phi}{\partial x}\\ \tilde{w}=\alpha\nabla_1^2F+\gamma\dfrac{\partial^2F}{\partial z^2}-\kappa^2\dfrac{\partial^2F}{\partial t^2}\end{cases}\tag{5}$$

式中，$\nabla_1^2=\dfrac{\partial^2}{\partial x^2}+\dfrac{\partial^2}{\partial y^2}$，$\alpha=2(1-\mu)$，$\gamma=1-2\mu$，$\kappa^2=(1-2\mu)\dfrac{\rho}{G}$。

位移函数应满足

$$\begin{gathered}\left(\nabla_1^2+\frac{\partial^2}{\partial z^2}-\frac{\rho}{G}\frac{\partial^2}{\partial t^2}\right)\left(\nabla_1^2+\frac{\partial^2}{\partial z^2}-\frac{1-2\mu}{2(1-\mu)}\frac{\rho}{G}\frac{\partial^2}{\partial t^2}\right)F=0\\ \nabla_1^2\Phi+\frac{\partial^2\Phi}{\partial z^2}-\frac{\rho}{G}\frac{\partial^2\Phi}{\partial t^2}=0\end{gathered}\tag{6}$$

对于简支矩形夹芯板，位移函数可假设为级数形式[23]：

$$\begin{aligned} F(x,y,z,t) &= \sum_{m}^{+\infty}\sum_{n}^{+\infty} f_{mn}(z) T_{mn}(x,y) \qquad m,n = 1,2,3,\ldots \\ \Phi(x,y,z,t) &= \sum_{m}^{+\infty}\sum_{n}^{+\infty} \varphi_{mn}(z) \frac{\partial^2 T_{mn}(x,y)}{\partial x \partial y} \qquad m,n = 1,2,3,\ldots \end{aligned} \tag{7}$$

式中，m、n为模态阶数。

面内因子$T_{mn}(x,y)$又可写为

$$\begin{aligned} T_{mn}(x,y) &= X_m(x) \cdot Y_n(y) \\ X_m(x) = \sin(\eta_m x), &\quad Y_n(y) = \sin(\nu_n y) \end{aligned} \tag{8}$$

根据简支边界条件，上式中$\eta_m = m\pi/a$，$\nu_m = n\pi/b$，a、b分别为矩形夹芯板的长和宽。将式(7)、(8)代入式(6)并化简得到

$$\begin{aligned} \left(\frac{\partial^2}{\partial z^2} + \frac{\rho\omega^2}{G} - k_{mn}{}^2\right)\left(\frac{\partial^2}{\partial z^2} + \frac{1-2\mu}{2(1-\mu)}\frac{\rho\omega^2}{G} - k_{mn}^2\right) f_{mn}(z) = 0 \\ \frac{\partial^2 \varphi_{mn}(z)}{\partial z^2} + \left(\frac{\rho\omega^2}{G} - k_{mn}^2\right)\varphi_{mn}(z) = 0 \end{aligned} \tag{9}$$

式中，$k_{mn}^2 = \eta_m^2 + \nu_n^2$。

式(9)有通解

$$\begin{aligned} \varphi_{mn}(z) &= A_{mn}\cosh\left(\lambda_{0_{mn}} k_{mn} z\right) + B_{mn}\sinh\left(\lambda_{0_{mn}} k_{mn} z\right) \\ f_{mn}(z) &= C_{mn}\cosh\left(\lambda_{1_{mn}} k_{mn} z\right) + D_{mn}\cosh\left(\lambda_{2_{mn}} k_{mn} z\right) + E_{mn}\sinh\left(\lambda_{1_{mn}} k_{mn} z\right) + F_{mn}\sinh\left(\lambda_{2_{mn}} k_{mn} z\right) \end{aligned} \tag{10}$$

式中，$\lambda_{0_{mn}} = \lambda_{1_{mn}} = \sqrt{1 - \frac{k_s^2}{k_{mn}^2}}$，$\lambda_{2_{mn}} = \sqrt{1 - \frac{k_l^2}{k_{mn}^2}}$，$k_s^2 = \frac{\rho\omega^2}{G}$，$k_l^2 = \frac{1-2\mu}{2(1-\mu)}\frac{\rho\omega^2}{G}$，$A_{mn}$、$B_{mn}$、$C_{mn}$、$D_{mn}$、$E_{mn}$、$F_{mn}$为待定系数。将式(7)、(10)代入式(5)得到芯材的位移为

$$\begin{cases} \tilde{u} = \sum_{m}^{+\infty}\sum_{n}^{+\infty} \left\{ \begin{aligned} &\nu_n^2\left[A_{mn}\cosh\left(\lambda_{0_{mn}} k_{mn} z\right) + B_{mn}\sinh\left(\lambda_{0_{mn}} k_{mn} z\right)\right] - \\ &\left[\begin{aligned} &C_{mn}\lambda_{1_{mn}} k_{mn}\sinh\left(\lambda_{1_{mn}} k_{mn} z\right) + D_{mn}\lambda_{2_{mn}} k_{mn}\sinh\left(\lambda_{2_{mn}} k_{mn} z\right) + \\ &E_{mn}\lambda_{1_{mn}} k_{mn}\cosh\left(\lambda_{1_{mn}} k_{mn} z\right) + F_{mn}\lambda_{2_{mn}} k_{mn}\cosh\left(\lambda_{2_{mn}} k_{mn} z\right) \end{aligned}\right] \end{aligned} \right\} X_m'(x) Y_n(y) \\ \tilde{v} = \sum_{m}^{+\infty}\sum_{n}^{+\infty} \left\{ \begin{aligned} &-\eta_m^2\left[A_{mn}\cosh\left(\lambda_{0_{mn}} k_{mn} z\right) + B_{mn}\sinh\left(\lambda_{0_{mn}} k_{mn} z\right)\right] - \\ &\left[\begin{aligned} &C_{mn}\lambda_{1_{mn}} k_{mn}\sinh\left(\lambda_{1_{mn}} k_{mn} z\right) + D_{mn}\lambda_{2_{mn}} k_{mn}\sinh\left(\lambda_{2_{mn}} k_{mn} z\right) + \\ &E_{mn}\lambda_{1_{mn}} k_{mn}\cosh\left(\lambda_{1_{mn}} k_{mn} z\right) + F_{mn}\lambda_{2_{mn}} k_{mn}\cosh\left(\lambda_{2_{mn}} k_{mn} z\right) \end{aligned}\right] \end{aligned} \right\} X_m(x) Y_n'(y) \\ \tilde{w} = \sum_{m}^{+\infty}\sum_{n}^{+\infty} \left\{ \begin{aligned} &C_{mn}\Gamma_{1_{mn}}\cosh\left(\lambda_{1_{mn}} k_{mn} z\right) + D_{mn}\Gamma_{2_{mn}}\cosh\left(\lambda_{2_{mn}} k_{mn} z\right) + \\ &E_{mn}\Gamma_{1_{mn}}\sinh\left(\lambda_{1_{mn}} k_{mn} z\right) + F_{mn}\Gamma_{2_{mn}}\sinh\left(\lambda_{2_{mn}} k_{mn} z\right) \end{aligned} \right\} X_m(x) Y_n(y) \end{cases} \tag{11}$$

式中，

$$\Gamma_{i_{mn}} = \gamma\left(\lambda_{i_{mn}}\right)^2 k_{mn}^2 + \omega^2\kappa^2 - \alpha k_{mn}^2 \qquad i = 1,2 \tag{12}$$

1.2 面板振动计算

令夹芯板的面板服从薄板假设，$\tilde{u}_1(x,y)$、$\tilde{v}_1(x,y)$、$\tilde{w}_1(x,y)$分别为面板1中性面的x、y、z方向的位

移，$\tilde{u}_2(x,y)$、$\tilde{v}_2(x,y)$、$\tilde{w}_2(x,y)$分别为面板2中性面的x、y、z方向的位移。同样可令

$$\tilde{u}_i=\sum_m^{+\infty}\sum_n^{+\infty}u_{i_{mn}}X_m'Y_n\quad,\quad\tilde{v}_i=\sum_m^{+\infty}\sum_n^{+\infty}v_{i_{mn}}X_mY_n'\quad,\quad\tilde{w}_i=\sum_m^{+\infty}\sum_n^{+\infty}w_{i_{mn}}X_mY_n\qquad m,n=1,2,3,\ldots\tag{13}$$

式中i =1、2，1表示面板1，2表示面板2。由于面板和芯材的界面处位移连续，因此有

$$\begin{cases}\tilde{u}\left(x,y,-\dfrac{h}{2}\right)=\tilde{u}_1(x,y)-\dfrac{t_1}{2}\dfrac{\partial\tilde{w}_1(x,y)}{\partial x}\quad,\quad\tilde{u}\left(x,y,\dfrac{h}{2}\right)=\tilde{u}_2(x,y)+\dfrac{t_2}{2}\dfrac{\partial\tilde{w}_2(x,y)}{\partial x}\\ \tilde{v}\left(x,y,-\dfrac{h}{2}\right)=\tilde{v}_1(x,y)-\dfrac{t_1}{2}\dfrac{\partial\tilde{w}_1(x,y)}{\partial y}\quad,\quad\tilde{v}\left(x,y,\dfrac{h}{2}\right)=\tilde{v}_2(x,y)+\dfrac{t_2}{2}\dfrac{\partial\tilde{w}_2(x,y)}{\partial y}\\ \tilde{w}\left(x,y,-\dfrac{h}{2}\right)=\tilde{w}_1(x,y)\quad,\quad\tilde{w}\left(x,y,\dfrac{h}{2}\right)=\tilde{w}_2(x,y)\end{cases}\tag{14}$$

式中，h为芯材的厚度。

将式(11)、(13)代入式(14)可以解得

$$\begin{aligned}
A_{mn}&=\frac{\left(u_{1_{mn}}+u_{2_{mn}}\right)-\left(v_{1_{mn}}+v_{2_{mn}}\right)}{2k_{mn}^2\cosh\left(R_{0_{mn}}\right)}\\
B_{mn}&=-\frac{\left(u_{1_{mn}}-u_{2_{mn}}\right)-\left(v_{1_{mn}}-v_{2_{mn}}\right)}{2k_{mn}^2\sinh\left(R_{0_{mn}}\right)}\\
C_{mn}&=\frac{\left\{\begin{array}{l}\Gamma_{2_{mn}}\left[2\eta_m^2\left(u_{1_{mn}}-u_{2_{mn}}\right)+\nu_n^2\left(v_{1_{mn}}-v_{2_{mn}}\right)-k_{mn}^2\left(t_1w_{1_{mn}}+t_2w_{2_{mn}}\right)\right]\cosh\left(R_{2_{mn}}\right)-\\ \quad 2\lambda_{2_{mn}}k_{mn}^3\left(w_{1_{mn}}+w_{2_{mn}}\right)\sinh\left(R_{2_{mn}}\right)\end{array}\right\}}{4k_{mn}^3M_{mn}}\\
D_{mn}&=-\frac{\left\{\begin{array}{l}\Gamma_{1_{mn}}\left[2\eta_m^2\left(u_{1_{mn}}-u_{2_{mn}}\right)+\nu_n^2\left(v_{1_{mn}}-v_{2_{mn}}\right)-k_{mn}^2\left(t_1w_{1_{mn}}+t_2w_{2_{mn}}\right)\right]\cosh\left(R_{1_{mn}}\right)-\\ \quad 2\lambda_{1_{mn}}k_{mn}^3\left(w_{1_{mn}}+w_{2_{mn}}\right)\sinh\left(R_{1_{mn}}\right)\end{array}\right\}}{4k_{mn}^3M_{mn}}\\
E_{mn}&=-\frac{\left\{\begin{array}{l}\Gamma_{2_{mn}}\left[2\eta_m^2\left(u_{1_{mn}}+u_{2_{mn}}\right)+\nu_n^2\left(v_{1_{mn}}+v_{2_{mn}}\right)-k_{mn}^2\left(t_1w_{1_{mn}}-t_2w_{2_{mn}}\right)\right]\sinh\left(R_{2_{mn}}\right)-\\ \quad 2\lambda_{2_{mn}}k_{mn}^3\left(w_{1_{mn}}-w_{2_{mn}}\right)\cosh\left(R_{2_{mn}}\right)\end{array}\right\}}{4k_{mn}^3N_{mn}}\\
F_{mn}&=\frac{\left\{\begin{array}{l}\Gamma_{1_{mn}}\left[2\eta_m^2\left(u_{1_{mn}}+u_{2_{mn}}\right)+\nu_n^2\left(v_{1_{mn}}+v_{2_{mn}}\right)-k_{mn}^2\left(t_1w_{1_{mn}}-t_2w_{2_{mn}}\right)\right]\sinh\left(R_{1_{mn}}\right)-\\ \quad 2\lambda_{1_{mn}}k_{mn}^3\left(w_{1_{mn}}-w_{2_{mn}}\right)\cosh\left(R_{1_{mn}}\right)\end{array}\right\}}{4k_{mn}^3N_{mn}}
\end{aligned}\tag{15}$$

式中，

$$R_{i_{mn}}=\frac{\lambda_{i_{mn}}k_{mn}h}{2}\qquad i=0,1,2$$

$$\Gamma_{i_{mn}}=\gamma\left(\lambda_{i_{mn}}\right)^2k_{mn}^2+\omega^2\kappa^2-\alpha k_{mn}^2\qquad i=1,2$$

$$M_{mn}=\Gamma_{2_{mn}}\lambda_{1_{mn}}\xi_{mn}-\Gamma_{1_{mn}}\lambda_{2_{mn}}\zeta_{mn}$$

$$N_{mn}=\Gamma_{2_{mn}}\lambda_{1_{mn}}\zeta_{mn}-\Gamma_{1_{mn}}\lambda_{2_{mn}}\xi_{mn}$$

$$\xi_{mn} = \sinh\left(R_{1_{mn}}\right)\cosh\left(R_{2_{mn}}\right) \tag{16}$$
$$\zeta_{mn} = \cosh\left(R_{1_{mn}}\right)\sinh\left(R_{2_{mn}}\right)$$
$$\chi_{mn} = \cosh\left(R_{1_{mn}}\right)\cosh\left(R_{2_{mn}}\right)$$
$$\vartheta_{mn} = \sinh\left(R_{1_{mn}}\right)\sinh\left(R_{2_{mn}}\right)$$

1.3 芯材和面板界面的应力

将式(11)代入式(2)可以求解得到芯材与面板界面的剪应力如下：

$$\tau_{xz}^{i} = G\sum_{m}^{+\infty}\sum_{n}^{+\infty}\left\{\begin{aligned}&\nu_n^2\lambda_{0_{mn}}k_{mn}\left\{A_{mn}\sinh\left[(-1)^i R_{0_{mn}}\right] + B_{mn}\cosh\left[(-1)^i R_{0_{mn}}\right]\right\} + \\ &C_{mn}\left[\Gamma_{1_{mn}} - \left(\lambda_{1_{mn}}\right)^2 k_{mn}^2\right]\cosh\left[(-1)^i R_{1_{mn}}\right] + \\ &D_{mn}\left[\Gamma_{2_{mn}} - \left(\lambda_{2_{mn}}\right)^2 k_{mn}^2\right]\cosh\left[(-1)^i R_{2_{mn}}\right] + \\ &E_{mn}\left[\Gamma_{1_{mn}} - \left(\lambda_{1_{mn}}\right)^2 k_{mn}^2\right]\sinh\left[(-1)^i R_{1_{mn}}\right] + \\ &F_{mn}\left[\Gamma_{2_{mn}} - \left(\lambda_{2_{mn}}\right)^2 k_{mn}^2\right]\sinh\left[(-1)^i R_{2_{mn}}\right]\end{aligned}\right\}X_m' Y_n \quad i = 1, 2 \tag{17}$$

$$\tau_{yz}^{i} = G\sum_{m}^{+\infty}\sum_{n}^{+\infty}\left\{\begin{aligned}&-\eta_m^2\lambda_{0_{mn}}k_{mn}\left\{A_{mn}\sinh\left[(-1)^i R_{0_{mn}}\right] + B_{mn}\cosh\left[(-1)^i R_{0_{mn}}\right]\right\} + \\ &C_{mn}\left[\Gamma_{1_{mn}} - \left(\lambda_{1_{mn}}\right)^2 k_{mn}^2\right]\cosh\left[(-1)^i R_{1_{mn}}\right] + \\ &D_{mn}\left[\Gamma_{2_{mn}} - \left(\lambda_{2_{mn}}\right)^2 k_{mn}^2\right]\cosh\left[(-1)^i R_{2_{mn}}\right] + \\ &E_{mn}\left[\Gamma_{1_{mn}} - \left(\lambda_{1_{mn}}\right)^2 k_{mn}^2\right]\sinh\left[(-1)^i R_{1_{mn}}\right] + \\ &F_{mn}\left[\Gamma_{2_{mn}} - \left(\lambda_{2_{mn}}\right)^2 k_{mn}^2\right]\sinh\left[(-1)^i R_{2_{mn}}\right]\end{aligned}\right\}X_m Y_n' \quad i = 1, 2 \tag{18}$$

式中，τ_{xz}^1、τ_{yz}^1、τ_{xz}^2、τ_{yz}^2分别表示界面1中x、y方向的剪力和界面2中x、y方向的剪力。同样可以解得芯材对面板的法向应力为

$$\sigma_z^i = \frac{2G}{1-2\mu}\sum_{m}^{+\infty}\sum_{n}^{+\infty}\left\{\begin{aligned}&C_{mn}\left[\mu k_{mn}^2 + (1-\mu)\Gamma_{1_{mn}}\right]\lambda_{1_{mn}}k_{mn}\sinh\left[(-1)^i R_{1_{mn}}\right] + \\ &D_{mn}\left[\mu k_{mn}^2 + (1-\mu)\Gamma_{2_{mn}}\right]\lambda_{2_{mn}}k_{mn}\sinh\left[(-1)^i R_{2_{mn}}\right] + \\ &E_{mn}\left[\mu k_{mn}^2 + (1-\mu)\Gamma_{1_{mn}}\right]\lambda_{1_{mn}}k_{mn}\cosh\left[(-1)^i R_{1_{mn}}\right] + \\ &F_{mn}\left[\mu k_{mn}^2 + (1-\mu)\Gamma_{2_{mn}}\right]\lambda_{2_{mn}}k_{mn}\cosh\left[(-1)^i R_{2_{mn}}\right]\end{aligned}\right\}X_m Y_n \quad i = 1, 2 \tag{19}$$

式中，σ_z^1、σ_z^2分别表示界面1中和界面2中芯材对面板的法向力。

2 简支矩形夹芯板振动运动方程

本章以夹芯板两侧面板的位移为未知数，获取夹芯板的振动运动方程。

2.1 简支矩形夹芯板面板法向振动运动方程

针对面板1的薄板微元，微元分别对x轴、y轴弯矩平衡，在z向力平衡。由于假设面板为薄板，故忽略面板的转动惯性力，仅计及法向振动的惯性力，因此有

$$\frac{\partial M_x^1}{\partial x}+\frac{\partial M_{xy}^1}{\partial y}=Q_x^1-\frac{t_1}{2}\tau_{xz}^1$$
$$\frac{\partial M_{xy}^1}{\partial x}+\frac{\partial M_y^1}{\partial y}=Q_y^1-\frac{t_1}{2}\tau_{yz}^1 \tag{20}$$
$$\frac{\partial Q_x^1}{\partial x}+\frac{\partial Q_y^1}{\partial y}+q_1=0$$

式中，上标"1"代表面板1，q_1为面板1微元受到的垂向力，包括外力、惯性力和芯材对面板的法向力，

$$q_1=P_1(x,y)-\rho_1 t_1\frac{\partial^2\tilde{w}_1}{\partial t^2}+\sigma_z^1 \tag{21}$$

式中，$P_1(x,y)$为作用在面板1的外载荷，ρ_1为面板1单位面积的质量，t_1为面板1的厚度。

对于面板1的薄板微元，有

$$\frac{\partial^2 M_x^1}{\partial x^2}+2\frac{\partial^2 M_{xy}^1}{\partial x\partial y}+\frac{\partial^2 M_y^1}{\partial y^2}=-D_1\nabla_1^2\nabla_1^2\tilde{w}_1 \tag{22}$$

式中，$\nabla_1^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$为二维的拉普拉斯算子；$D_1=\frac{E_1 t_1^3}{12\left(1-\mu_1^2\right)}$ 为面板1的抗弯刚度；E_1、μ_1分别为面板1的杨氏模量和泊松比。

由式(20)、(21)、(22)得到

$$D_1\nabla_1^2\nabla_1^2\tilde{w}_1+\rho_1 t_1\frac{\partial^2\tilde{w}_1}{\partial t^2}=P_1(x,y)+\sigma_z^1+\frac{t_1}{2}\left(\frac{\partial\tau_{xz}^1}{\partial x}+\frac{\partial\tau_{yz}^1}{\partial y}\right) \tag{23}$$

将外载荷表示为$P_1(x,y)=\sum_m^{+\infty}\sum_n^{+\infty}p_{1_{mn}}X_m Y_n\mathrm{e}^{\mathrm{j}\omega t}$，并将式(17)、(18)、(19)代入式(23)，比较每一个级数项的系数，整理得到的面板1的第(m,n)阶振动运动方程为

$$D_1 k_{mn}^4 w_{1_{mn}}-\omega^2\rho_1 t_1 w_{1_{mn}}+\frac{G}{2k_{mn}^3}\left[\begin{array}{l}\eta_m^2\left(\frac{J_{1_{mn}}}{M_{mn}}+\frac{H_{1_{mn}}}{N_{mn}}\right)u_{1_{mn}}+\eta_m^2\left(-\frac{J_{1_{mn}}}{M_{mn}}+\frac{H_{1_{mn}}}{N_{mn}}\right)u_{2_{mn}}+\\ \nu_n^2\left(\frac{J_{1_{mn}}}{M_{mn}}+\frac{H_{1_{mn}}}{N_{mn}}\right)v_{1_{mn}}+\nu_n^2\left(-\frac{J_{1_{mn}}}{M_{mn}}+\frac{H_{1_{mn}}}{N_{mn}}\right)v_{2_{mn}}-\\ \frac{1}{2}k_{mn}^2\left(\frac{t_1 J_{1_{mn}}+2k_{mn}I_{1_{mn}}}{M_{mn}}+\frac{t_1 H_{1_{mn}}+2k_{mn}K_{1_{mn}}}{N_{mn}}\right)w_{1_{mn}}-\\ \frac{1}{2}k_{mn}^2\left(\frac{t_2 J_{1_{mn}}+2k_{mn}I_{1_{mn}}}{M_{mn}}-\frac{t_2 H_{1_{mn}}+2k_{mn}K_{1_{mn}}}{N_{mn}}\right)w_{2_{mn}}\end{array}\right]=p_{1_{mn}} \tag{24}$$

同理，面板2的第(m,n)阶振动运动方程为

$$D_2 k_{mn}^4 w_{2_{mn}}-\omega^2\rho_2 t_2 w_{2_{mn}}+\frac{G}{2k_{mn}^3}\left[\begin{array}{l}\eta_m^2\left(\frac{J_{2_{mn}}}{M_{mn}}-\frac{H_{2_{mn}}}{N_{mn}}\right)u_{1_{mn}}-\eta_m^2\left(\frac{J_{2_{mn}}}{M_{mn}}+\frac{H_{2_{mn}}}{N_{mn}}\right)u_{2_{mn}}+\\ \nu_n^2\left(\frac{J_{2_{mn}}}{M_{mn}}-\frac{H_{2_{mn}}}{N_{mn}}\right)v_{1_{mn}}-\nu_n^2\left(\frac{J_{2_{mn}}}{M_{mn}}+\frac{H_{2_{mn}}}{N_{mn}}\right)v_{2_{mn}}-\\ \frac{k_{mn}^2}{2}\left[t_1\left(\frac{J_{2_{mn}}}{M_{mn}}-\frac{H_{2_{mn}}}{N_{mn}}\right)+2k_{mn}\left(\frac{I_{2_{mn}}}{M_{mn}}-\frac{K_{2_{mn}}}{N_{mn}}\right)\right]w_{1_{mn}}-\\ \frac{k_{mn}^2}{2}\left[t_2\left(\frac{J_{2_{mn}}}{M_{mn}}+\frac{H_{2_{mn}}}{N_{mn}}\right)+2k_{mn}\left(\frac{I_{2_{mn}}}{M_{mn}}+\frac{K_{2_{mn}}}{N_{mn}}\right)\right]w_{2_{mn}}\end{array}\right]=p_{2_{mn}} \tag{25}$$

式中，ρ_2为面板2单位面积的质量，t_2为面板2的厚度。

$$
\begin{aligned}
&D_i=\frac{E_i t_i^3}{12\left(1-\mu_i^2\right)} \quad i=1,2\\
&p_{i_{mn}}=\frac{4}{ab}\int_0^a\int_0^b P_i(x,y)X_mY_n\mathrm{d}x\mathrm{d}y \quad i=1,2\\
&J_{i_{mn}}=\Gamma_{2_{mn}}\left(\Pi_{1_{mn}}\lambda_{1_{mn}}k_{mn}\xi_{mn}+t_i\Omega_{1_{mn}}\chi_{mn}\right)-\Gamma_{1_{mn}}\left(\Pi_{2_{mn}}\lambda_{2_{mn}}k_{mn}\zeta_{mn}+t_i\Omega_{2_{mn}}\chi_{mn}\right) \quad i=1,2\\
&H_{i_{mn}}=\Gamma_{2_{mn}}\left(\Pi_{1_{mn}}\lambda_{1_{mn}}k_{mn}\zeta_{mn}+t_i\Omega_{1_{mn}}\vartheta_{mn}\right)-\Gamma_1^{mn}\left(\Pi_{2_{mn}}\lambda_{2_{mn}}k_{mn}\xi_{mn}+t_i\Omega_{2_{mn}}\vartheta_{mn}\right) \quad i=1,2\\
&I_{i_{mn}}=\lambda_{2_{mn}}\left(\Pi_{1_{mn}}\lambda_{1_{mn}}k_{mn}\vartheta_{mn}+t_i\Omega_{1_{mn}}\zeta_{mn}\right)-\lambda_{1_{mn}}\left(\Pi_{2_{mn}}\lambda_{2_{mn}}k_{mn}\vartheta_{mn}+t_i\Omega_{2_{mn}}\xi_{mn}\right) \quad i=1,2\\
&K_{i_{mn}}=\lambda_{2_{mn}}\left(\Pi_{1_{mn}}\lambda_{1_{mn}}k_{mn}\chi_{mn}+t_i\Omega_{1_{mn}}\xi_{mn}\right)-\lambda_{1_{mn}}\left(\Pi_{2_{mn}}\lambda_{2_{mn}}k_{mn}\chi_{mn}+t_i\Omega_{2_{mn}}\zeta_{mn}\right) \quad i=1,2\\
&\Omega_{i_{mn}}=\frac{k_{mn}^2}{2}\left[\Gamma_{i_{mn}}-\left(\lambda_{i_{mn}}\right)^2k_{mn}^2\right] \quad i=1,2\\
&\Pi_{i_{mn}}=\frac{2}{1-2\mu}\left[\mu k_{mn}^2+(1-\mu)\Gamma_{i_{mn}}\right] \quad i=1,2
\end{aligned} \tag{26}
$$

2.2 简支矩形夹芯板面板面内振动运动方程

面板1微元在x方向的振动运动方程如下：

$$t_1\left(\frac{\partial\sigma_x^1}{\partial x}+\frac{\partial\tau_{xy}^1}{\partial y}\right)=\rho_1t_1\frac{\partial^2\tilde{u}_1}{\partial t^2}-\tau_{xz}^1 \tag{27}$$

由于面板1为薄板，因此有

$$
\begin{aligned}
\sigma_x^1(x,y)&=\frac{E_1}{1-\mu_1^2}\left(\frac{\partial\tilde{u}_1}{\partial x}+\mu_1\frac{\partial\tilde{v}_1}{\partial y}\right)\\
\tau_{xy}^1(x,y)&=G_1\left(\frac{\partial\tilde{u}_1}{\partial y}+\frac{\partial\tilde{v}_1}{\partial x}\right)
\end{aligned} \tag{28}
$$

将式(17)、(28)代入式(27)，可以得到面板1在x方向第(m,n)阶振动运动方程为

$$
\begin{aligned}
&-t_1\left[\left(\frac{\eta_m^2E_1}{1-\mu_1^2}+\nu_n^2G_1\right)u_{1_{mn}}+\nu_n^2\left(\frac{\mu_1E_1}{1-\mu_1^2}+G_1\right)v_{1_{mn}}\right]+\omega^2\rho_1t_1u_{1_{mn}}+\\
&G\left[\begin{array}{l}
-\left[\frac{\eta_m^2}{2k_{mn}}R_{mn}+\frac{\nu_n^2\lambda_{0_{mn}}}{k_{mn}\tanh\left(2R_{0_{mn}}\right)}\right]u_{1_{mn}}+\left[\frac{\eta_m^2}{2k_{mn}}S_{mn}+\frac{\nu_n^2\lambda_{0_{mn}}}{k_{mn}\sinh\left(2R_{0_{mn}}\right)}\right]u_{2_{mn}}-\\
\left[\frac{\nu_n^2}{2k_{mn}}R_{mn}-\frac{\nu_n^2\lambda_{0_{mn}}}{k_{mn}\tanh\left(2R_{0_{mn}}\right)}\right]v_{1_{mn}}+\left[\frac{\nu_n^2}{2k_{mn}}S_{mn}-\frac{\nu_n^2\lambda_{0_{mn}}}{k_{mn}\sinh\left(2R_{0_{mn}}\right)}\right]v_{2_{mn}}-\\
\frac{1}{2}\left[\begin{array}{l}\lambda_{2_{mn}}\left(\frac{\zeta_{mn}}{M_{mn}}+\frac{\xi_{mn}}{N_{mn}}\right)\left[\Gamma_{1_{mn}}-\left(\lambda_{1_{mn}}\right)^2k_{mn}^2\right]-\\ \lambda_{1_{mn}}\left(\frac{\xi_{mn}}{M_{mn}}+\frac{\zeta_{mn}}{N_{mn}}\right)\left[\Gamma_{2_{mn}}-\left(\lambda_{2_{mn}}\right)^2k_{mn}^2\right]-\frac{t_1k_{mn}}{2}R_{mn}\end{array}\right]w_{1_{mn}}-\\
\frac{1}{2}\left[\begin{array}{l}\lambda_{2_{mn}}\left(\frac{\zeta_{mn}}{M_{mn}}-\frac{\xi_{mn}}{N_{mn}}\right)\left[\Gamma_{1_{mn}}-\left(\lambda_{1_{mn}}\right)^2k_{mn}^2\right]-\\ \lambda_{1_{mn}}\left(\frac{\xi_{mn}}{M_{mn}}-\frac{\zeta_{mn}}{N_{mn}}\right)\left[\Gamma_{2_{mn}}-\left(\lambda_{2_{mn}}\right)^2k_{mn}^2\right]-\frac{t_2k_{mn}}{2}S_{mn}\end{array}\right]w_{2_{mn}}
\end{array}\right]=0
\end{aligned} \tag{29}
$$

同理,面板1在y方向、面板2在x方向和面板2在y方向第(m,n)阶的振动运动方程分别为

$$
\begin{aligned}
&-t_1\left[\left(\frac{\nu_n^2E_1}{1-\mu_1^2}+\eta_m^2G_1\right)v_{1_{mn}}+\eta_m^2\left(\frac{\mu_1E_1}{1-\mu_1^2}+G_1\right)u_{1_{mn}}\right]+\omega^2\rho_1t_1v_{1_{mn}}+\\
&G\left\{\begin{array}{l}
\left[-\dfrac{\eta_m^2R_{mn}}{2k_{mn}}+\dfrac{\eta_m^2\lambda_{0_{mn}}}{k_{mn}\tanh\left(2R_{0_{mn}}\right)}\right]u_{1_{mn}}+\left[\dfrac{\eta_m^2S_{mn}}{2k_{mn}}-\dfrac{\eta_m^2\lambda_{0_{mn}}}{k_{mn}\sinh\left(2R_{0_{mn}}\right)}\right]u_{2_{mn}}-\\
\left[\dfrac{\nu_n^2R_{mn}}{2k_{mn}}+\dfrac{\eta_m^2\lambda_{0_{mn}}}{k_{mn}\tanh\left(2R_{0_{mn}}\right)}\right]v_{1_{mn}}+\left[\dfrac{\nu_n^2S_{mn}}{2k_{mn}}+\dfrac{\eta_m^2\lambda_{0_{mn}}}{k_{mn}\sinh\left(2R_{0_{mn}}\right)}\right]v_{2_{mn}}-\\
\dfrac{1}{2}\left[\begin{array}{l}\lambda_{2_{mn}}\left(\dfrac{\zeta_{mn}}{M_{mn}}+\dfrac{\xi_{mn}}{N_{mn}}\right)\left[\Gamma_{1_{mn}}-\left(\lambda_{1_{mn}}\right)^2k_{mn}^2\right]-\\ \lambda_{1_{mn}}\left(\dfrac{\xi_{mn}}{M_{mn}}+\dfrac{\zeta_{mn}}{N_{mn}}\right)\left[\Gamma_{2_{mn}}-\left(\lambda_{2_{mn}}\right)^2k_{mn}^2\right]-\dfrac{k_{mn}t_1}{2}R_{mn}\end{array}\right]w_{1_{mn}}-\\
\dfrac{1}{2}\left[\begin{array}{l}\lambda_{2_{mn}}\left(\dfrac{\zeta_{mn}}{M_{mn}}-\dfrac{\xi_{mn}}{N_{mn}}\right)\left[\Gamma_{1_{mn}}-\left(\lambda_{1_{mn}}\right)^2k_{mn}^2\right]-\\ \lambda_{1_{mn}}\left(\dfrac{\xi_{mn}}{M_{mn}}-\dfrac{\zeta_{mn}}{N_{mn}}\right)\left[\Gamma_{2_{mn}}-\left(\lambda_{2_{mn}}\right)^2k_{mn}^2\right]-\dfrac{k_{mn}t_2}{2}S_{mn}\end{array}\right]w_{2_{mn}}
\end{array}\right\}=0
\end{aligned}
\tag{30}
$$

$$
\begin{aligned}
&-t_2\left[\left(\frac{\eta_m^2E_2}{1-\mu_2^2}+\nu_n^2G_2\right)u_{2_{mn}}+\nu_n^2\left(\frac{\mu_2E_2}{1-\mu_2^2}+G_2\right)v_{2_{mn}}\right]+\omega^2\rho_2t_2u_{2_{mn}}=\\
&G\left\{\begin{array}{l}
-\left[\dfrac{\eta_m^2S_{mn}}{2k_{mn}}+\dfrac{\nu_n^2\lambda_{0_{mn}}}{k_{mn}\sinh\left(2R_{0_{mn}}\right)}\right]u_{1_{mn}}+\left[\dfrac{\eta_m^2R_{mn}}{2k_{mn}}+\dfrac{\nu_n^2\lambda_{0_{mn}}}{k_{mn}\tanh\left(2R_{0_{mn}}\right)}\right]u_{2_{mn}}-\\
\left[\dfrac{\nu_n^2S_{mn}}{2k_{mn}}-\dfrac{\nu_n^2\lambda_{0_{mn}}}{k_{mn}\sinh\left(2R_{0_{mn}}\right)}\right]v_{1_{mn}}+\left[\dfrac{\nu_n^2R_{mn}}{2k_{mn}}-\dfrac{\nu_n^2\lambda_{0_{mn}}}{k_{mn}\tanh\left(2R_{0_{mn}}\right)}\right]v_{2_{mn}}-\\
\dfrac{1}{2}\left[\begin{array}{l}\lambda_{2_{mn}}\left(\dfrac{\zeta_{mn}}{M_{mn}}-\dfrac{\xi_{mn}}{N_{mn}}\right)\left[\Gamma_{1_{mn}}-\left(\lambda_{1_{mn}}\right)^2k_{mn}^2\right]-\\ \lambda_{1_{mn}}\left(\dfrac{\xi_{mn}}{M_{mn}}-\dfrac{\zeta_{mn}}{N_{mn}}\right)\left[\Gamma_{2_{mn}}-\left(\lambda_{2_{mn}}\right)^2k_{mn}^2\right]-\dfrac{t_1k_{mn}}{2}S_{mn}\end{array}\right]w_{1_{mn}}-\\
\dfrac{1}{2}\left[\begin{array}{l}\lambda_{2_{mn}}\left(\dfrac{\zeta_{mn}}{M_{mn}}+\dfrac{\xi_{mn}}{N_{mn}}\right)\left[\Gamma_{1_{mn}}-\left(\lambda_{1_{mn}}\right)^2k_{mn}^2\right]-\\ \lambda_{1_{mn}}\left(\dfrac{\xi_{mn}}{M_{mn}}+\dfrac{\zeta_{mn}}{N_{mn}}\right)\left[\Gamma_{2_{mn}}-\left(\lambda_{2_{mn}}\right)^2k_{mn}^2\right]-\dfrac{t_2k_{mn}}{2}R_{mn}\end{array}\right]w_{2_{mn}}
\end{array}\right\}
\end{aligned}
\tag{31}
$$

$$-t_2\left[\left(\frac{\nu_n^2E_2}{1-\mu_2^2}+\eta_m^2G_2\right)v_{2_{mn}}+\eta_m^2\left(\frac{\mu_2E_2}{1-\mu_2^2}+G_2\right)u_{2_{mn}}\right]+\omega^2\rho_2t_2v_{2_{mn}}=$$

$$G\left\{\begin{array}{l}-\left[\dfrac{\eta_m^2S_{mn}}{2k_{mn}}-\dfrac{\eta_m^2\lambda_{0_{mn}}}{k_{mn}\sinh\left(2R_{0_{mn}}\right)}\right]u_{1_{mn}}+\left[\dfrac{\eta_m^2R_{mn}}{2k_{mn}}-\dfrac{\eta_m^2\lambda_{0_{mn}}}{k_{mn}\tanh\left(2R_{0_{mn}}\right)}\right]u_{2_{mn}}-\\ \left[\dfrac{\nu_n^2S_{mn}}{2k_{mn}}+\dfrac{\eta_m^2\lambda_{0_{mn}}}{k_{mn}\sinh\left(2R_{0_{mn}}\right)}\right]v_{1_{mn}}+\left[\dfrac{\nu_n^2R_{mn}}{2k_{mn}}+\dfrac{\eta_m^2\lambda_{0_{mn}}}{k_{mn}\tanh\left(2R_{0_{mn}}\right)}\right]v_{2_{mn}}-\\ \dfrac{1}{2}\left[\begin{array}{l}\lambda_{2_{mn}}\left(\dfrac{\zeta_{mn}}{M_{mn}}-\dfrac{\xi_{mn}}{N_{mn}}\right)\left[\Gamma_{1_{mn}}-\left(\lambda_{1_{mn}}\right)^2k_{mn}^2\right]-\\ \lambda_{1_{mn}}\left(\dfrac{\xi_{mn}}{M_{mn}}-\dfrac{\zeta_{mn}}{N_{mn}}\right)\left[\Gamma_{2_{mn}}-\left(\lambda_{2_{mn}}\right)^2k_{mn}^2\right]-\dfrac{t_1k_{mn}}{2}S_{mn}\end{array}\right]w_{1_{mn}}-\\ \dfrac{1}{2}\left[\begin{array}{l}\lambda_{2_{mn}}\left(\dfrac{\zeta_{mn}}{M_{mn}}+\dfrac{\xi_{mn}}{N_{mn}}\right)\left[\Gamma_{1_{mn}}-\left(\lambda_{1_{mn}}\right)^2k_{mn}^2\right]-\\ \lambda_{1_{mn}}\left(\dfrac{\xi_{mn}}{M_{mn}}+\dfrac{\zeta_{mn}}{N_{mn}}\right)\left[\Gamma_{2_{mn}}-\left(\lambda_{2_{mn}}\right)^2k_{mn}^2\right]-\dfrac{t_2k_{mn}}{2}R_{mn}\end{array}\right]w_{2_{mn}}\end{array}\right\}\tag{32}$$

2.3 简支矩形夹芯板振动运动方程求解

方程(24)、(25)、(29)~(32)是6个以$u_{1_{mn}}$、$v_{1_{mn}}$、$w_{1_{mn}}$、$u_{2_{mn}}$、$v_{2_{mn}}$、$w_{2_{mn}}$为未知数的方程，六个未知数分别是面板1和面板2在x、y、z三个方向的位移，这六个方程即为简支矩形夹芯板的振动运动方程组。将这六个方程联立，便可求解夹芯板第(m,n)阶模态的两个面板的面外和面内振动。为便于求解，可将六个控制方程写成矩阵的表达形式：

$$\boldsymbol{L}_{mn}\cdot\boldsymbol{d}_{mn}=\boldsymbol{P}_{mn}\tag{33}$$

式中，$\boldsymbol{P}_{mn}=\left\{p_{1_{mn}}\quad p_{2_{mn}}\quad 0\quad 0\quad 0\quad 0\right\}^{\mathrm{T}}$，为外载荷组成的向量；$\boldsymbol{d}_{mn}=\left\{u_{1_{mn}}\quad v_{1_{mn}}\quad w_{1_{mn}}\quad u_{2_{mn}}\quad v_{2_{mn}}\quad w_{2_{mn}}\right\}^{\mathrm{T}}$，为上下面板位移组成的向量，也是待求的未知数；$\boldsymbol{L}_{mn}$为6×6的矩阵，其中的元素如附录A所示。

当简支矩形夹芯板进行自由振动时，外载荷向量$\boldsymbol{P}_{mn}$所有元素为0。为了在位移向量$\boldsymbol{d}_{mn}$不为零向量的同时保证式(33)成立，需要$\left|\boldsymbol{L}_{mn}\right|=0$，可通过此式求解简支矩形夹芯板第$(m,n)$阶的固有频率。

当简支矩形夹芯板受到外载荷作用进行受迫振动时，可通过式(26)中的第一式计算获得外载荷向量$\boldsymbol{P}_{mn}$，然后对式(33)作矩阵运算，获得简支矩形夹芯板两侧面板共6个模态位移的值，结合式(13)获得两侧面板的位移。再将计算得到的6个面板模态位移值代入式(11)、(15)，即可获得相应芯材的振动位移。

3 简支矩形夹芯板振动固有频率计算

为验证上一章推导的振动运动方程的正确性，现选取一个简支矩形夹芯板为算例，求解其自由振动的固有频率，并与其他夹芯板理论和有限元计算结果进行对比验证。该简支矩形夹芯板长和宽分别为0.6 m、0.5 m，面板1和面板2厚度均为5 mm，夹芯厚度为100 mm，面板和夹芯的材料和属性如表1所示。

表1 材料力学性能

Tab.1 Material properties

	材料	模量/GPa	泊松比	密度/(kg·m^{-3})
面板	玻璃钢	20	0.15	2000
夹芯	PVC泡沫	0.055	0.32	100

同时，为了对理论方法进行比较和评价，采用有限元仿真结果作为参考基准。用有限元软件Abaqus 6.14建立该夹芯板的有限元模型，对其固有频率进行计算。有限元模型的芯材采用三维实体单元，面板采用壳单元。为更好地模拟芯材膨胀和剪切并存的振动行为，在夹芯板厚度方向划分多层网格，且取二阶单元进行计算，模型如图4所示。表2为本文理论、Reissner理论[1]、Mead理论[3]的计算结果分别与有限元计算结果的对比。

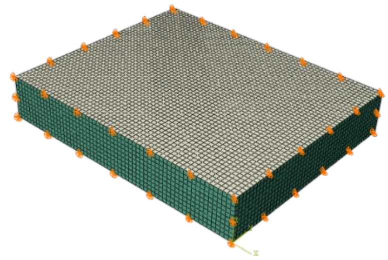


图4 夹芯板有限元模型

Fig.4 Finite element model of sandwich panel

表2 夹芯板固有频率

Tab.2 Natural frequencies of sandwich panel

阶数	模态振型	有限元计算频率/Hz	Reissner理论[1]		Mead理论[3]		本文理论	
			频率/Hz	误差	频率/Hz	误差	频率/Hz	误差
1	(1,1)	346.6	333.9	-3.7%	352.0	1.6%	350.6	1.2%
2	(2,1)	531.6	506.0	-4.8%	538.1	1.2%	534.1	0.5%
3	(1,2)	595.8	565.4	-5.1%	603.4	1.3%	597.8	0.3%
4	(2,2)	721.2	681.4	-5.5%	732.7	1.6%	722.9	0.2%
5	(3,1)	747.3	705	-5.7%	759.5	1.6%	748.6	0.2%
6	(1,3)	869.4	816.6	-6.1%	887.2	2.0%	870.5	0.1%
7	(3,2)	894.9	839.9	-6.1%	914.3	2.2%	896.1	0.1%
8	(2,3)	962.3	900.8	-6.4%	985.8	2.4%	963.6	0.1%
9	(4,1)	976.2	913.6	-6.4%	1000.9	2.5%	977.8	0.2%
10	(4,2)	1097.4	1021.2	-6.9%	1130.1	3.0%	1098.7	0.1%

表2中，Reissner理论只考虑夹芯板面板的抗拉压能力及芯材的抗剪能力，并假设夹芯板在振动中没有法向形变，因此其计算误差较大，且误差随模态阶数的升高(频率升高)变大。Mead的理论考虑了面板本身的抗弯能力及夹芯板芯材与面板的相互作用，但同样假设夹芯板在振动中没有法向形变，其计算结果的误差比Reissner理论小，但依然随模态阶数的升高变大，可见忽略夹芯板法向变形的假定在中高频段并不适用。本文计算方法在高模态阶数时依然可保持稳定的计算精度，不仅可以证明本文方法对简支矩形夹芯板的固有频率预报的有效性，同时也证明了“面板为薄板，芯材为三维弹性体”的假定适用于夹芯板中高频段的振动预报。

4 简支矩形夹芯板受迫振动响应计算

在上述夹芯板算例的面板1的中心施加单位激励力，直接采用式(33)求解激励点的振动加速度响应，加速度的参考级为1×10^{-6} m/s^2。依然以有限元计算结果为参考基准，把本文理论、Reissner理论、Mead理论的计算结果和有限元计算结果进行对比，如图5所示。

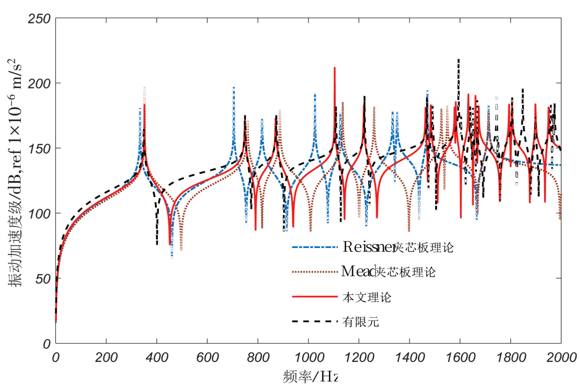


图5 夹芯板受迫振动计算结果

Fig.5 Calculation results of forced vibration of sandwich panel

由图5可见，在1~1000 Hz频段，Mead理论、本文理论和有限元计算结果的前几阶振动峰所在频点十分接近，且均为弯曲振动模式(以首阶振动峰为例，振型用本文理论计算，如图6所示，夹芯板面板1和面板2振动位移方向一致)，而Reissner理论的振动峰所在频点偏低，且偏差随频率升高而增大，这是由于Reissner理论仅考虑

夹芯板面板的抗拉压能力及芯材的抗剪能力，在理论计算中夹芯板的刚度偏小而导致。在1000~2000 Hz频段，本文理论依旧与有限元计算结果吻合良好，且比Reissner理论和Mead理论多了数个振动峰。经分析，多出的峰值多数反映出膨胀振动模式的影响(以本文理论计算的1583 Hz处的振动峰为例，振型如图7所示，夹芯板面板1和面板2振动位移方向相反)，而由于Reissner理论和Mead理论忽略夹芯板法向变形，因而不能体现夹芯板膨胀振动导致的响应部分。这进一步证明了在中高频段的振动预报中，对芯材应用三维弹性力学方法是必要的，但面板应用薄板理论也可保持较高的精度。

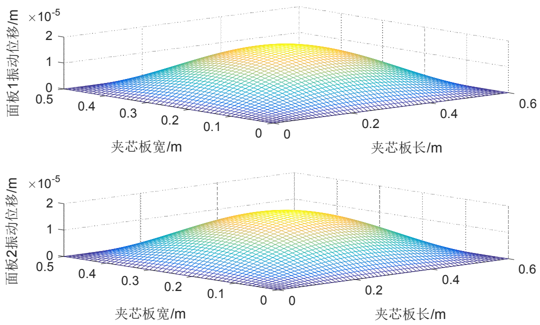


图6 典型弯曲振动振型(350 Hz)

Fig.6 Typical bending vibration mode (350 Hz)

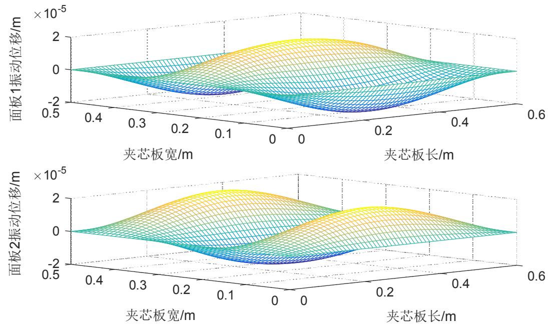


图7 典型膨胀振动振型(1583 Hz)

Fig.7 Typical dilatational vibration mode (1583 Hz)

同时可以看出，在此算例中，膨胀振动模式在1000 Hz起始的频段就会对夹芯板振动的计算结果造成明显影响，若不考虑膨胀振动模式的影响，会导致预报中出现较大的误差。在抑振隔声结构应用中，膨胀振动模式也会大大降低夹芯板在该频段的抑振隔声性能，因此膨胀振动模式的影响在夹芯板的抑振隔声设计时也应纳入考虑，不应忽略。

5 结　语

本文假设夹芯板面板为薄板，芯材为三维弹性体，提出一种新的夹芯板振动计算方法，可对夹芯板的固有频率和振动响应进行计算。本文方法容许夹芯板在振动中存在法向形变，综合体现夹芯板弯曲振动模式和膨胀振动模式的影响。通过算例将本文理论方法的计算结果与其他理论计算以及有限元计算的结果进行了对比，证明了本文理论方法的有效性和优势。本文提出的计算方法可为中高频段夹芯板的预报和设计提供支撑。

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附录A：夹芯板振动运动方程中矩阵L_{mn}的元素

$$L_{11}=\frac{\eta_m^2 G}{2k_{mn}^3}\left(\frac{J_{1_{mn}}}{M_{mn}}+\frac{H_{1_{mn}}}{N_{mn}}\right)$$

$$L_{12}=\frac{\nu_n^2 G}{2k_{mn}^3}\left(\frac{J_{1_{mn}}}{M_{mn}}+\frac{H_{1_{mn}}}{N_{mn}}\right)$$

$$L_{13}=D_1k_{mn}^4-\omega^2\rho_1t_1-\frac{G}{4k_{mn}}\left(\frac{t_1J_{1_{mn}}+2k_{mn}I_{1_{mn}}}{M_{mn}}+\frac{t_1H_{1_{mn}}+2k_{mn}K_{1_{mn}}}{N_{mn}}\right)$$

$$L_{14}=\frac{\eta_m^2 G}{2k_{mn}^3}\left(-\frac{J_{1_{mn}}}{M_{mn}}+\frac{H_{1_{mn}}}{N_{mn}}\right)$$

$$L_{15}=\frac{\nu_n^2 G}{2k_{mn}^3}\left(-\frac{J_{1_{mn}}}{M_{mn}}+\frac{H_{1_{mn}}}{N_{mn}}\right)$$

$$L_{16}=-\frac{G}{4k_{mn}}\left(\frac{t_2J_{1_{mn}}+2k_{mn}I_{1_{mn}}}{M_{mn}}-\frac{t_2H_{1_{mn}}+2k_{mn}K_{1_{mn}}}{N_{mn}}\right)$$

$$L_{21}=\frac{\eta_m^2 G}{2k_{mn}^3}\left(\frac{J_{2_{mn}}}{M_{mn}}-\frac{H_{2_{mn}}}{N_{mn}}\right)$$

$$L_{22}=\frac{\nu_n^2 G}{2k_{mn}^3}\left(\frac{J_{2_{mn}}}{M_{mn}}-\frac{H_{2_{mn}}}{N_{mn}}\right)$$

$$L_{23}=-\frac{G}{4k_{mn}}\left[t_1\left(\frac{J_{2_{mn}}}{M_{mn}}-\frac{H_{2_{mn}}}{N_{mn}}\right)+2k_{mn}\left(\frac{I_{2_{mn}}}{M_{mn}}-\frac{K_{2_{mn}}}{N_{mn}}\right)\right]$$

$$L_{24}=-\frac{\eta_m^2 G}{2k_{mn}^3}\left(\frac{J_{2_{mn}}}{M_{mn}}+\frac{H_{2_{mn}}}{N_{mn}}\right)$$

$$L_{25}=-\frac{\nu_n^2 G}{2k_{mn}^3}\left(\frac{J_{2_{mn}}}{M_{mn}}+\frac{H_{2_{mn}}}{N_{mn}}\right)$$

$$L_{26}=D_2k_{mn}^4-\omega^2\rho_2t_2-\frac{G}{4k_{mn}}\left[t_2\left(\frac{J_{2_{mn}}}{M_{mn}}+\frac{H_{2_{mn}}}{N_{mn}}\right)+2k_{mn}\left(\frac{I_{2_{mn}}}{M_{mn}}+\frac{K_{2_{mn}}}{N_{mn}}\right)\right]$$

$$L_{31}=-G\left[\frac{\eta_m^2R_{mn}}{2k_{mn}}+\frac{\nu_n^2\lambda_{0_{mn}}}{k_{mn}\tanh\left(2R_{0_{mn}}\right)}\right]-t_1\left(\frac{\eta_m^2E_1}{1-\mu_1^2}+\nu_n^2G_1\right)+\omega^2\rho_1t_1$$

$$L_{32}=-G\left[\frac{\nu_n^2R_{mn}}{2k_{mn}}-\frac{\nu_n^2\lambda_{0_{mn}}}{k_{mn}\tanh\left(2R_{0_{mn}}\right)}\right]-t_1\nu_n^2\left(\frac{\mu_1E_1}{1-\mu_1^2}+G_1\right)$$

$$L_{33}=-\frac{G}{2}\left[\lambda_{2_{mn}}\left(\frac{\zeta_{mn}}{M_{mn}}+\frac{\xi_{mn}}{N_{mn}}\right)\left[\Gamma_{1_{mn}}-\left(\lambda_{1_{mn}}\right)^2k_{mn}^2\right]-\lambda_{1_{mn}}\left(\frac{\xi_{mn}}{M_{mn}}+\frac{\zeta_{mn}}{N_{mn}}\right)\left[\Gamma_{2_{mn}}-\left(\lambda_{2_{mn}}\right)^2k_{mn}^2\right]-\frac{t_1k_{mn}}{2}R_{mn}\right]$$

$$L_{34}=G\left[\frac{\eta_m^2S_{mn}}{2k_{mn}}+\frac{\nu_n^2\lambda_{0_{mn}}}{k_{mn}\sinh\left(2R_{0_{mn}}\right)}\right]$$

$$L_{35}=G\left[\frac{\nu_n^2 S_{mn}}{2k_{mn}}-\frac{\nu_n^2\lambda_{0_{mn}}}{k_{mn}\sinh\left(2R_{0_{mn}}\right)}\right]$$

$$L_{36}=-\frac{G}{2}\left[\lambda_{2_{mn}}\left(\frac{\zeta_{mn}}{M_{mn}}-\frac{\xi_{mn}}{N_{mn}}\right)\left[\Gamma_{1_{mn}}-\left(\lambda_{1_{mn}}\right)^2k_{mn}^2\right]-\lambda_{1_{mn}}\left(\frac{\xi_{mn}}{M_{mn}}-\frac{\zeta_{mn}}{N_{mn}}\right)\left[\Gamma_{2_{mn}}-\left(\lambda_{2_{mn}}\right)^2k_{mn}^2\right]-\frac{t_2k_{mn}}{2}S_{mn}\right]$$

$$L_{41}=G\left[-\frac{\eta_m^2 R_{mn}}{2k_{mn}}+\frac{\eta_m^2\lambda_{0_{mn}}}{k_{mn}\tanh\left(2R_{0_{mn}}\right)}\right]-t_1\eta_m^2\left(\frac{\mu_1E_1}{1-\mu_1^2}+G_1\right)$$

$$L_{42}=-G\left[\frac{\nu_n^2 R_{mn}}{2k_{mn}}+\frac{\eta_m^2\lambda_{0_{mn}}}{k_{mn}\tanh\left(2R_{0_{mn}}\right)}\right]-t_1\left(\frac{\nu_n^2E_1}{1-\mu_1^2}+\eta_m^2G_1\right)+\omega^2\rho_1t_1$$

$$L_{43}=-\frac{G}{2}\left[\lambda_{2_{mn}}\left(\frac{\zeta_{mn}}{M_{mn}}+\frac{\xi_{mn}}{N_{mn}}\right)\left[\Gamma_{1_{mn}}-\left(\lambda_{1_{mn}}\right)^2k_{mn}^2\right]-\lambda_{1_{mn}}\left(\frac{\xi_{mn}}{M_{mn}}+\frac{\zeta_{mn}}{N_{mn}}\right)\left[\Gamma_{2_{mn}}-\left(\lambda_{2_{mn}}\right)^2k_{mn}^2\right]-\frac{k_{mn}t_1}{2}R_{mn}\right]$$

$$L_{44}=G\left[\frac{\eta_m^2 S_{mn}}{2k_{mn}}-\frac{\eta_m^2\lambda_{0_{mn}}}{k_{mn}\sinh\left(2R_{0_{mn}}\right)}\right]$$

$$L_{45}=G\left[\frac{\nu_n^2 S_{mn}}{2k_{mn}}+\frac{\eta_m^2\lambda_{0_{mn}}}{k_{mn}\sinh\left(2R_{0_{mn}}\right)}\right]$$

$$L_{46}=-\frac{G}{2}\left[\lambda_{2_{mn}}\left(\frac{\zeta_{mn}}{M_{mn}}-\frac{\xi_{mn}}{N_{mn}}\right)\left[\Gamma_{1_{mn}}-\left(\lambda_{1_{mn}}\right)^2k_{mn}^2\right]-\lambda_{1_{mn}}\left(\frac{\xi_{mn}}{M_{mn}}-\frac{\zeta_{mn}}{N_{mn}}\right)\left[\Gamma_{2_{mn}}-\left(\lambda_{2_{mn}}\right)^2k_{mn}^2\right]-\frac{k_{mn}t_2}{2}S_{mn}\right]$$

$$L_{51}=G\left[\frac{\eta_m^2 S_{mn}}{2k_{mn}}+\frac{\nu_n^2\lambda_{0_{mn}}}{k_{mn}\sinh\left(2R_{0_{mn}}\right)}\right]$$

$$L_{52}=G\left[\frac{\nu_n^2 S_{mn}}{2k_{mn}}-\frac{\nu_n^2\lambda_{0_{mn}}}{k_{mn}\sinh\left(2R_{0_{mn}}\right)}\right]$$

$$L_{53}=\frac{G}{2}\left[\lambda_{2_{mn}}\left(\frac{\zeta_{mn}}{M_{mn}}-\frac{\xi_{mn}}{N_{mn}}\right)\left[\Gamma_{1_{mn}}-\left(\lambda_{1_{mn}}\right)^2k_{mn}^2\right]-\lambda_{1_{mn}}\left(\frac{\xi_{mn}}{M_{mn}}-\frac{\zeta_{mn}}{N_{mn}}\right)\left[\Gamma_{2_{mn}}-\left(\lambda_{2_{mn}}\right)^2k_{mn}^2\right]-\frac{t_1k_{mn}}{2}S_{mn}\right]$$

$$L_{54}=-G\left[\frac{\eta_m^2 R_{mn}}{2k_{mn}}+\frac{\nu_n^2\lambda_{0_{mn}}}{k_{mn}\tanh\left(2R_{0_{mn}}\right)}\right]-t_2\left(\frac{\eta_m^2E_2}{1-\mu_2^2}+\nu_n^2G_2\right)+\omega^2\rho_2t_2$$

$$L_{55}=-G\left[\frac{\nu_n^2 R_{mn}}{2k_{mn}}-\frac{\nu_n^2\lambda_{0_{mn}}}{k_{mn}\tanh\left(2R_{0_{mn}}\right)}\right]-t_2\nu_n^2\left(\frac{\mu_2E_2}{1-\mu_2^2}+G_2\right)$$

$$L_{56}=\frac{G}{2}\left[\lambda_{2_{mn}}\left(\frac{\zeta_{mn}}{M_{mn}}+\frac{\xi_{mn}}{N_{mn}}\right)\left[\Gamma_{1_{mn}}-\left(\lambda_{1_{mn}}\right)^2k_{mn}^2\right]-\lambda_{1_{mn}}\left(\frac{\xi_{mn}}{M_{mn}}+\frac{\zeta_{mn}}{N_{mn}}\right)\left[\Gamma_{2_{mn}}-\left(\lambda_{2_{mn}}\right)^2k_{mn}^2\right]-\frac{t_2k_{mn}}{2}R_{mn}\right]$$

$$L_{61}=G\left[\frac{\eta_m^2 S_{mn}}{2k_{mn}}-\frac{\eta_m^2\lambda_{0_{mn}}}{k_{mn}\sinh\left(2R_{0_{mn}}\right)}\right]$$

$$L_{62} = G\left[\frac{\nu_n^2 S_{mn}}{2k_{mn}} + \frac{\eta_m^2 \lambda_{0_{mn}}}{k_{mn}\sinh\left(2R_{0_{mn}}\right)}\right]$$

$$L_{63} = \frac{G}{2}\left[\lambda_{2_{mn}}\left(\frac{\zeta_{mn}}{M_{mn}} - \frac{\xi_{mn}}{N_{mn}}\right)\left[\Gamma_{1_{mn}} - \left(\lambda_{1_{mn}}\right)^2 k_{mn}^2\right] - \lambda_{1_{mn}}\left(\frac{\xi_{mn}}{M_{mn}} - \frac{\zeta_{mn}}{N_{mn}}\right)\left[\Gamma_{2_{mn}} - \left(\lambda_{2_{mn}}\right)^2 k_{mn}^2\right] - \frac{t_1 k_{mn}}{2} S_{mn}\right]$$

$$L_{64} = -G\left[\frac{\eta_m^2 R_{mn}}{2k_{mn}} - \frac{\eta_m^2 \lambda_{0_{mn}}}{k_{mn}\tanh\left(2R_{0_{mn}}\right)}\right] - t_2 \eta_m^2\left(\frac{\mu_2 E_2}{1 - \mu_2^2} + G_2\right)$$

$$L_{65} = -G\left[\frac{\nu_n^2 R_{mn}}{2k_{mn}} + \frac{\eta_m^2 \lambda_{0_{mn}}}{k_{mn}\tanh\left(2R_{0_{mn}}\right)}\right] - t_2\left(\frac{\nu_n^2 E_2}{1 - \mu_2^2} + \eta_m^2 G_2\right) + \omega^2 \rho_2 t_2$$

$$L_{66} = \frac{G}{2}\left[\lambda_{2_{mn}}\left(\frac{\zeta_{mn}}{M_{mn}} + \frac{\xi_{mn}}{N_{mn}}\right)\left[\Gamma_{1_{mn}} - \left(\lambda_{1_{mn}}\right)^2 k_{mn}^2\right] - \lambda_{1_{mn}}\left(\frac{\xi_{mn}}{M_{mn}} + \frac{\zeta_{mn}}{N_{mn}}\right)\left[\Gamma_{2_{mn}} - \left(\lambda_{2_{mn}}\right)^2 k_{mn}^2\right] - \frac{t_2 k_{mn}}{2} R_{mn}\right]$$